Hybrid All-Pay and Winner-Pay Contests

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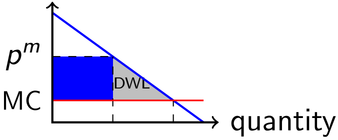
Introduction: Contests (1/3)

- **Contests** are common in economic, social and political life:
  - sports, military combat, war;
  - political compet'n, rent-seeking for rents allocated by regulator;
  - marketing, advertising, patent races, relative reward schemes in firms, beauty contests between firms, litigation.
- A common modeling approach:
  - Contestant i chooses $x_i \geq 0$ to max $\pi_i = v_i p_i \left(x_1, x_2, \ldots x_n\right) - x_i$ where p_i is a differentiable contest success funct. ($p_i = \frac{x_i^r}{\sum_{j=1}^n x_j^r}$).
- Gordon Tullock's motivation for studying the dissipation rent:
  - Empirical studies in the 1950s: DWL appears to be tiny.
  - Tullock: Maybe a part of profits adds to the cost of monopoly.



Introduction: Hybrid contest (2/3)

- A **hybrid contest**:
  - In some contests, each contestant can make both **all-pay investments** and **winner-pay investments**.
- Example: The competitive bidding to host the Olympic games.
  - *All-pay investments*: Candidate cities spend money upfront, with the goal of persuading members of the IOC.
  - *Winner-pay investments*: A city commits to build new stadia and invest in safety arrangements if being awarded the Games.
- To fix ideas, consider the following formalization:
  - Contestant i chooses $x_i \geq 0$ and $y_i \geq 0$ to maximize

$$\pi_i = (v_i - y_i)\, p_i\, (s_1, s_2, \ldots s_n) - x_i,$$

subject to $s_i = f\,(x_i, y_i)$.

Introduction: Other examples (3/3)

Further examples

- Competition for a government contract or grant:
  - *All-pay investments*: Time/effort spent on preparing proposal.
  - *Winner-pay investments*: Commit to ambitious customer service.
- A political election:
  - *All-pay investments*: Campaign expenditures.
  - *Winner-pay investments*: Electoral promises (costly if they deviate from the politician's own ideal policy).
- Rent seeking to win monopoly rights of a regulated market:
  - *All-pay investments*: Ex ante bribes (how Tullock modeled it).
  - *Winner-pay investments*: Conditional bribes.

Literature review (1/1)

- Two earlier papers that model a hybrid contest:
  - **Haan and Schonbeek (2003)**.
    - They assume Cobb-Douglas—which here is quite restrictive.
  - **Melkonyan (2013)**.
    - CES but with $\sigma \geq 1$. Symmetric model. Hard to check SOC.
    - My analysis: (i) other approach which yields easy-to-check existence condition; (ii) assumes general production function and CSF; (iii) studies both symmetric and asymmetric models.

- Other contest models with more than one influence channel:
  - **Sabotage in contests** (improve own performance and sabotage the others' performance): Konrad (2000), Chen (2003).
  - **War and conflict** (choice of production and appropriation): Hirschleifer (1991) and Skaperdas and Syroploulos (1997).
  - **Multiple all-pay "arms"** (maybe with different costs): Arbatskaya and Mialon (2010).

A model of a hybrid contest (1/3)

- $n \geq 2$ contestants try to win an indivisible prize.
- Contestant i chooses $x_i \geq 0$ and $y_i \geq 0$ to maximize the following payoff:

$$\pi_i = (v_i - y_i)\, p_i(\mathbf{s}) - x_i, \qquad \text{subject to } s_i = f(x_i, y_i),$$

where $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ and $s_i \geq 0$ is contestant i's *score*.

  - $v_i > 0$ is i's valuation of the prize.
  - $p_i(\mathbf{s})$ is i's prob. of winning (or contest success function, CSF).
  - x_i is the **all-pay investment**: paid whether i wins or not.
  - y_i is the **winner-pay investment**: paid i.f.f. i wins.
- It is a one-shot game where the contestants choose their investments (x_i, y_i) simultaneously with each other.

A model of a hybrid contest (2/3)

Assumptions about the production function $f(x_i, y_i)$

- Thrice continuously differentiable in its arguments.
- Strictly increasing in each of its arguments.
- Strictly quasiconcave.
- Homogeneous of degree $t > 0$: $\forall k > 0 \; f(kx_i, ky_i) = k^t f(x_i, y_i)$.
- Satisfies $f(0, 0) = 0$.
- Inada conditions to rule out $x_i = 0$ or $y_i = 0$.
- Example (CES):

$$f(x_i, y_i) = \left[\alpha x_i^{\frac{\sigma-1}{\sigma}} + (1-\alpha) y_i^{\frac{\sigma-1}{\sigma}}\right]^{\frac{t\sigma}{\sigma-1}}, \qquad \alpha \in (0,1), \sigma > 0$$

A model of a hybrid contest (3/3)

Assumptions about the contest success function $p_i(\mathbf{s})$

$$p_i(\mathbf{s}) \in [0,1], \quad \text{with} \quad \sum_{i=1}^{n} p_i(\mathbf{0}) \leq 1 \ \text{ and } \ \sum_{i=1}^{n} p_i(\mathbf{s}) = 1 \text{ for all } \mathbf{s} \neq \mathbf{0},$$

- Twice continuously differentiable for all $\mathbf{s} \in \Re_+^n \setminus \{\mathbf{0}\}$.
- Strictly increasing and strictly concave in s_i.
- Strictly decreasing in s_j for all $j \neq i$.
- If $s_i = 0$ and $s_j > 0$ for some $j \neq i$, then $p_i(\mathbf{s}) = 0$.
- Any values of $p_i(\mathbf{0}) \leq 1$ allowed, although $p_i(\mathbf{0}) < 1$ for all i.
- Later I assume that $p_i(\mathbf{s})$ is homogeneous in $\mathbf{s}$.
- Example (extended Tullock):

$$p_i(\mathbf{s}) = \frac{w_i s_i^r}{\sum_{j=1}^{n} w_j s_j^r}, \qquad w_i, r > 0.$$

Analysis (1/7)

- One possible approach:
  - Plug the production function into the CSF.
  - Take FOCs w.r.t. x_i and y_i.
  - Used by Haan and Schoonbeek (2003) and Melkonyan (2013), assuming Cobb-Douglas and CES, respectively.
- My approach: Solve for contestant i's best reply in two steps:
  1. Compute the conditional factor demands.
     - That is, derive optimal x_i and y_i, given $\mathbf{s}$ (so also given s_i).
  2. Plug the factor demands into the payoff and then characterize contestant i's optimal score s_i (given $\mathbf{s_{-i}}$).
- Important advantage: a single choice variable at 2, so easier to determine what conditions are required for equilibrium existence.

- Contestant i solves (for fixed p_i): $\min_{x_i,y_i} p_i y_i + x_i$, subject to $f(x_i, y_i) = s_i$.
- The first-order conditions (λ_i is the Lagrange multiplier):

$$\frac{\partial \mathcal{L}_i}{\partial x_i} = 1 - \lambda_i f_1(x_i, y_i) = 0, \qquad \frac{\partial \mathcal{L}_i}{\partial y_i} = p_i - \lambda_i f_2(x_i, y_i) = 0.$$

- So, by combining the FOCs:

$$\frac{1}{p_i} = \frac{f_1(x_i, y_i)}{f_2(x_i, y_i)} \overset{\text{def}}{=} g\left(\frac{x_i}{y_i}\right) \Rightarrow x_i = y_i h\left(\frac{1}{p_i}\right),$$

  where h is the inverse of g (i.e., $h \overset{\text{def}}{=} g^{-1}$).
- By plugging back into $s_i = f(x_i, y_i)$ and rewriting, we obtain:

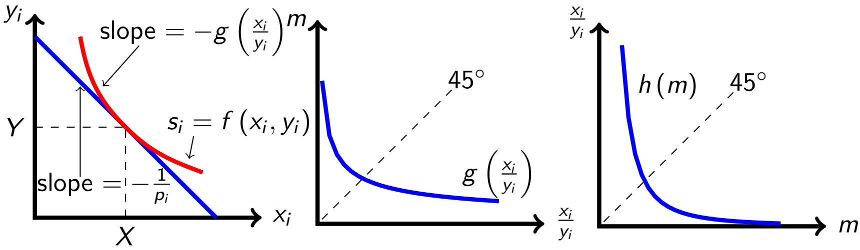
$$Y_i(s_i, p_i) = \left[\frac{s_i}{f(h(1/p_i), 1)}\right]^{\frac{1}{t}}, \quad X_i(s_i, p_i) = Y_i(s_i, p_i)\, h\left(\frac{1}{p_i}\right).$$

- Contestant i's payoff: $\pi_i(\mathbf{s}) = p_i(\mathbf{s}) v_i - C_i[s_i, p_i(\mathbf{s})]$, where

$$C_i[s_i, p_i(\mathbf{s})] \overset{\text{def}}{=} p_i(\mathbf{s})\, Y_i[s_i, p_i(\mathbf{s})] + X_i[s_i, p_i(\mathbf{s})].$$

- A Nash equilibrium of the hybrid contest:
  - A profile $\mathbf{s}^*$ such that $\pi_i(\mathbf{s}^*) \geq \pi_i\left(s_i, \mathbf{s}^*_{-\mathbf{i}}\right)$, all i and all $s_i \geq 0$.

The cost-minimization problem and the h function



(a) Cost minimization.

(b) Graph of the g function.

(c) Graph of the h function.

Analysis (4/7)

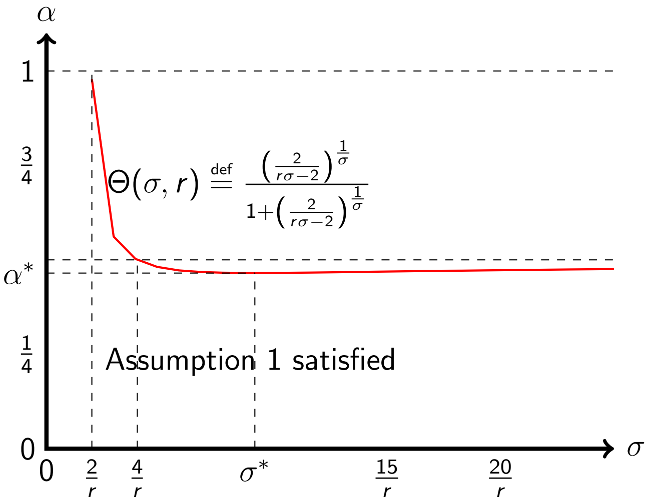
Equilibrium existence

Define the following elasticities:

- The elasticity of output w.r.t. x_i: $\eta\left(\frac{1}{p_i}\right) \overset{\text{def}}{=} \frac{f_1\left[h\left(\frac{1}{p_i}\right),1\right]h\left(\frac{1}{p_i}\right)}{f\left[h\left(\frac{1}{p_i}\right),1\right]}$.
- The elasticity of substitution: $\sigma\left(\frac{1}{p_i}\right) \overset{\text{def}}{=} -\frac{h'\left(\frac{1}{p_i}\right)\frac{1}{p_i}}{h\left(\frac{1}{p_i}\right)}$.
- The elasticity of the win probability w.r.t. s_i: $\varepsilon_i(\mathbf{s}) \overset{\text{def}}{=} \frac{\partial p_i}{\partial s_i}\frac{s_i}{p_i}$.
- We have that $\eta \in (0,t)$, $\sigma > 0$, and $\varepsilon_i \in (0,1)$.
- **Assumption 1.** The production function and the CSF satisfy:
$$t \leq 1 \text{ and } \varepsilon_i(\mathbf{s})\,\eta\left(\frac{1}{p_i}\right)\sigma\left(\frac{1}{p_i}\right) \leq 2 \quad (\text{for all } p_i \text{ and } \mathbf{s}).$$
- **Proposition 1.** Suppose Assumption 1 is satisfied. Then there exists a pure strategy Nash equilibrium of the hybrid contest.

- Assume a CES production function, $t = 1$, $r \leq 1$, and

$$p_i(\mathbf{s}) = \frac{w_i s_i^r}{\sum_{j=1}^n w_j s_j^r} \quad \text{and} \quad p_i(0, \cdots, 0) = \frac{w_i}{\sum_{j=1}^n w_j}.$$



Analysis (6/7)

- To check the SOC with Melkonyan's analytical approach is cumbersome and in the end he relies on numerical simulations:

[. . .] one can demonstrate, after a series of tedious algebraic manipulations, that a player's payoff function is locally concave at the symmetric equilibrium candidate in (7) if and only if [large mathematical expression].

[. . .] Numerical simulations indicate that this inequality is violated only for extreme values of the parameters [. . .].

[. . .] In addition to verifying the local second-order conditions, I have used numerical simulations to verify that the global second-order conditions are satisfied under a wide range of scenarios.

Characterization of equilibrium

- Recall: $\pi_i(\mathbf{s}) = p_i(\mathbf{s}) v_i - C_i[s_i, p_i(\mathbf{s})]$.
- The FOC (with an equality if $s_i > 0$):

$$\frac{\partial \pi_i(\mathbf{s})}{\partial s_i} = \frac{\partial p_i(\mathbf{s})}{\partial s_i} v_i - C_1(s_i, p_i) - C_2(s_i, p_i) \frac{\partial p_i(\mathbf{s})}{\partial s_i} \leq 0.$$

- Use Shephard's lemma, $C_2(s_i, p_i) = Y_i[s_i, p_i(\mathbf{s})]$:

$$[v_i - Y_i(s_i, p_i(\mathbf{s}))] \frac{\partial p_i(\mathbf{s})}{\partial s_i} \leq C_1(s_i, p_i), \tag{1}$$

  with an equality if $s_i > 0$.
- **Proposition 2.** Suppose Assumption 1 is satisfied. Then $\mathbf{s}^* = (s_1^*, \ldots, s_n^*)$ is a pure strategy Nash equilibrium of the hybrid contest if and only if condition (1) holds, with equality if $s_i^* > 0$, for each contestant i. Moreover, $\mathbf{s} = \mathbf{0}$ is not a Nash equilibrium.

A Symmetric Hybrid Contest (1/4)

Assumption 2. The CSF is symmetric and homogeneous of degree 0.

- Note that, thanks to Assumption 2:

$$\frac{\partial p_i(s, s, \ldots, s)}{\partial s_i} = \frac{\widehat{\varepsilon}(n)}{ns}, \text{ where } \widehat{\varepsilon}(n) \stackrel{\text{def}}{=} \varepsilon_i(1, 1, \ldots, 1).$$

- Use this in the FOC and impose symmetry:

$$(v - y^*) \frac{\widehat{\varepsilon}(n)}{ns^*} = C_1 \left[s^*, \frac{1}{n} \right] = \frac{1}{ts^*} C \left[s^*, \frac{1}{n} \right] = \frac{1}{ts^*} \left[\frac{y^*}{n} + x^* \right]$$

$\Leftrightarrow (v - y^*)\, t\widehat{\varepsilon}(n) = y^* + nx^*$. And from before, $x^* = h(n)y^*$.

- The last equalities are linear in x^* and y^*, so easy to solve.
- **Proposition 3.** Within the family of sym. eq., there is a unique pure strategy equilibrium: $s^* = f[h(n), 1]\,(y^*)^t$, $x^* = h(n)y^*$, and

$$y^* = \frac{t\widehat{\varepsilon}(n)v}{1 + nh(n) + t\widehat{\varepsilon}(n)}.$$

- **Proposition 4.** Effect of more contestants on x^* and y^*:

$$\frac{\partial x^*}{\partial n} < 0 \Leftrightarrow \sigma(n) > -\frac{n\,(n-2)\,h(n) - 1}{(n-1)\,[1 + t\widehat{\varepsilon}(n)]},$$

$$\frac{\partial y^*}{\partial n} > 0 \Leftrightarrow \sigma(n) > \frac{n(n-2)h(n) - 1}{(n-1)nh(n)};$$

  and if $\sigma(n) \geq 1$, then necessarily $\frac{\partial x^*}{\partial n} < 0$ and $\frac{\partial y^*}{\partial n} > 0$.
- In order to understand the above:
  - More contestants means a lower probability of winning.
  - This lowers the relative cost of investing in y_i.
  - So whenever $\sigma(n)$ is sufficiently large, $\frac{\partial y^*}{\partial n} > 0$ and $\frac{\partial x^*}{\partial n} < 0$.
  - But if $\sigma(n)$ small, the derivatives must have the same sign. For:

$$\frac{\partial y^*}{\partial n}\frac{n}{y^*} = \sigma(n) + \frac{\partial x^*}{\partial n}\frac{n}{x^*} \qquad \text{(follows from } x^* = h(n)y^*\text{)}.$$

    As $\sigma(n) \to 0$, the production function requires x_i and y_i to be used in fixed proportions (a Leontief production technology).

- The total amount of equilibrium expenditures in the symmetric hybrid model is defined as $R^{\mathrm{H}} \stackrel{\mathrm{def}}{=} nC\left[s^*, \frac{1}{n}\right]$.
- The corresponding amount in the all-pay contest: $R^{\mathrm{A}} = t\widehat{\varepsilon}(n)v$.
- **Proposition 5, part (a).** In the symmetric model:

$$R^{\mathrm{H}} = (1 - \frac{y^*}{v})R^{\mathrm{A}} = \left[\frac{1}{v\left[1 + nh(n)\right]} + \frac{1}{R^{\mathrm{A}}}\right]^{-1}.$$

In particular, for any finite n, we have $R^{\mathrm{H}} < R^{\mathrm{A}}$.

  - The payoff suggests the intuition: $\pi_i = (v_i - y_i)\, p_i\left(\mathbf{s}\right) - x_i$.
- **Proposition 5, part (b).** In the symmetric model, suppose $p_i(\mathbf{s}) = s_i^r / \sum_{j=1}^{n} s_j^r$, with $r > 0$.
  - Then R^{H} is weakly increasing in n if and only if: (i)

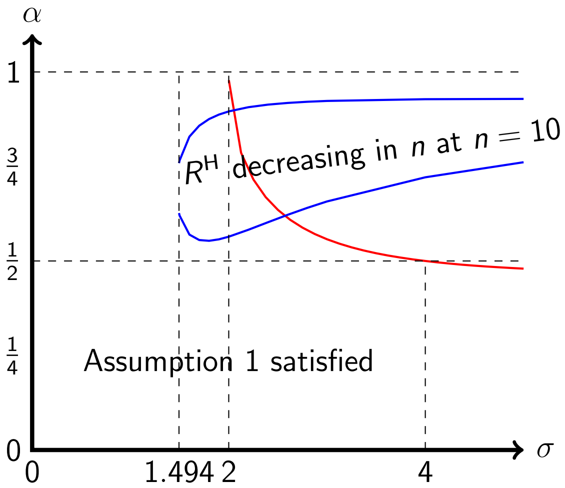
$$\sigma\left(n\right) \leq 1 + \frac{4n}{tr\left(n-1\right)^2}; \qquad (2)$$

or (ii) inequality (2) is violated and $h\left(n\right) \notin \left(\Xi_L, \Xi_H\right)$. See figure!

A Symmetric Hybrid Contest (4/4)

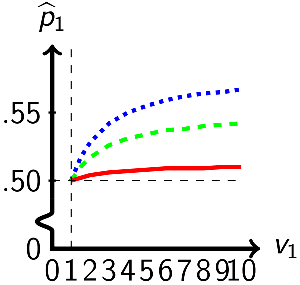
Illustration of result (b)

- Assume CES, $t = 1$, and $n = 10$.

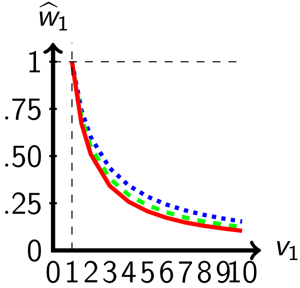


Asym. hybrid contest with endogenous bias

- Two contestants. Different valuations. CSF potentially biased.
- Cobb-Douglas prod. f. and extended Tullock CSF.
- A principal chooses the bias to max. total expenditures.
- Result: High-valuation contestant more likely to win but the bias is against her (the latter might not be robust).



(a) The high-valuation contestant's probability of winning.



(b) The weight in the CSF that is assigned to the high-valuation contestant's score.

Main results and contributions (1/1)

1. The analytical approach (borrowing from producer theory):
   - $\longrightarrow$ Generality, tractability, and an existence condition.

2. A larger n leads to substitution away from all-pay investments.
   - But only if the elasticity of substitution is large enough.

3. Total expenditures always lower in hybrid contest than in all-pay.

4. T. exp. can be decreasing in n (also shown by Melkonyan).

5. Asym. contests (in terms of valuations and bias): Predictions about relative size of investments and of expenditures.

6. Endogenous bias: High-valuation contestant more likely to win but the bias is against her (the latter might not be robust).

Possible avenues for future work (1/1)

1. Sequential moves: first (x_1, y_1), then (x_2, y_2).
   - Strategic complements/substitutes depending on whether $\varepsilon_i(\mathbf{s})\,\eta\left(\frac{1}{p_i}\right)\sigma\left(\frac{1}{p_i}\right) \gtrless 1$.
2. Risk averse contestants.
3. Applications to other contests with multiple influence channels.
   - Limitation: only s_i, not x_i and y_i directly, matter for outcome.
4. Experimental testing. (Relatively sharp predictions. But risk neutrality might be an issue? Hard to vary σ in lab?)
5. Further work on asymmetric contests.
6. Contest design in broader settings.

Asymmetric Hybrid Contests (1/3)

- I assume $n = 2$ and I study three models:
  - The CSF is biased in favor of one contestant.
  - One contestant has a higher valuation than the other.
  - I also endogenize the degree of bias.
- **Assumption 3.** The CSF is given by

$$p_i(\mathbf{s}) = \frac{w_i s_i^r}{w_1 s_1^r + w_2 s_2^r}.$$

- The following three equations define equilibrium values of p_1^*, y_1^*, and y_2^*:

$$y_i^* = \frac{rtp_i^*(1-p_i^*)v_i}{rtp_i^*(1-p_i^*) + p_i^* + h\left(\frac{1}{p_i^*}\right)}, \quad \text{for } i = 1, 2, \text{ and } \Upsilon(p_1^*) = 0, \text{ where}$$

$$\Upsilon(p_1) \stackrel{\text{def}}{=} \frac{\frac{w_2 v_2^{rt}}{w_1 v_1^{rt}} p_1 f\left[h\left(\frac{1}{1-p_1}\right), 1\right]^r}{\left[rtp_1(1-p_1) + 1 - p_1 + h\left(\frac{1}{1-p_1}\right)\right]^{rt}} - \frac{(1-p_1) f\left[h\left(\frac{1}{p_1}\right), 1\right]^r}{\left[rtp_1(1-p_1) + p_1 + h\left(\frac{1}{p_1}\right)\right]^{rt}}.$$

- The equilibrium is unique if $r\eta\left(\frac{1}{p_i}\right)\sigma\left(\frac{1}{p_i}\right) \leq 1$.

Asymmetric Hybrid Contests (2/3)

A Biased decision process $(w_1 \neq w_2$ **but** $v_1 = v_2)$

- Among the results:
  - **(a)** $p_1^* > p_2^* \Leftrightarrow y_1^* < y_2^* \Leftrightarrow C(s_1^*, p_1^*) > C(s_2^*, p_2^*)$.
  - **(b)** Evaluated at symmetry $(w_1 = w_2)$: $\frac{\partial p_1^*}{\partial w_1} > 0$,

$$\frac{\partial y_1^*}{\partial w_1} < 0, \quad \frac{\partial y_2^*}{\partial w_1} > 0, \quad \frac{\partial x_1^*}{\partial w_1} > 0 \Leftrightarrow \frac{\partial x_2^*}{\partial w_1} < 0 \Leftrightarrow \sigma(2) > \frac{2}{2+rt}.$$

Different valuations $(v_1 \neq v_2$ **but** $w_1 = w_2)$

- Among the results:
  - **(a)** $p_1^* > p_2^* \Leftrightarrow \frac{y_1^*}{v_1} < \frac{y_2^*}{v_2}$.
  - **(b)** $v_1 - y_1^* > v_2 - y_2^* \Leftrightarrow C(s_1^*, p_1^*) > C(s_2^*, p_2^*)$.

An Endogenous Bias (w_1 chosen, but $v_1 \geq v_2$ and w_2 fixed)

- Timing of events in the game:
  1. A principal chooses w_1 to maximize $R^H = C(s_1^*, p_1^*) + C(s_2^*, p_2^*)$.
  2. w_1 becomes common knowledge and the contestants interact as in the previous analysis.
- **Assumption 3.** The production function is of Cobb-Douglas form: $f(x_i, y_i) = x_i^{\alpha} y_i^{\beta}$, for $\alpha > 0$ and $\beta > 0$.
- Results: The equilibrium values of p_1 and w_1 satisfy:
  - If $v_1 = v_2$, then $\widehat{p}_1 = \frac{1}{2}$ and $\widehat{w}_1 = w_2$.
  - If $v_1 > v_2$, then $\widehat{p}_1 > \frac{1}{2}$.
  - If $v_1 > v_2$, then $\widehat{w}_1 < w_2$ at least if $|v_1 - v_2|$ is very small or big.
- My intuition for results:
  - Contestant 1 is more valuable as a contributor (as $v_1 > v_2$).
  - Hence, she should be encouraged to use x_1, as all-pay investments are more conducive to large expenditures.
  - This is achieved by making winner-pay inv. costly: $\widehat{p}_1 > \frac{1}{2}$.
  - To generate $\widehat{p}_1 > \frac{1}{2}$, $v_1 > v_2$ is more than enough, so bias can be in favor of Contestant 2.
    - Might not be robust.

Literature review (2/2)

- Multidimensional (procurement) auctions:
  - **Che (2003), Branck (1997), Asker and Cantillon (2008)**.
    - Firms bid on both price and (many dimensions of) quality.
    - The components of each bid jointly determine a score.
    - Auctioneer chooses bidder with highest score.
  - Differences:
    - In their models, not both all-pay and winner-pay ingredients.
    - Not a probabilistic CSF.
- Optimal design of a research contest: **Che and Gale (2003)**.
  - A principal wants to procure an innovation.
  - Fimrs choose both quality of innovation and the prize if winning.
  - Thus, effectively, both all-pay and winner-pay ingredients.
  - Differences: Not a probabilistic CSF (so mixed strategy eq.), linear production function, mechanism design approach.