Hybrid All-Pay and Winner-Pay Contests
Seminar at DICE in Düsseldorf, June 5, 2018

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June 2, 2018
A **hybrid contest**: 
- In some economic, social, or political situation, each one of a number of economic agents try to win an indivisible prize.
- To increase her probability of winning, each contestant makes both **all-pay investments** and **winner-pay investments**.

**Example**: The competitive bidding to host the Olympic games.
- **All-pay investments**: Candidate cities spend money upfront, with the goal of persuading members of the IOC.
- **Winner-pay investments**: A city commits to build new stadia and invest in safety arrangements if being awarded the Games.

To fix ideas, consider the following formalization:
- Contestant $i$ chooses $x_i \geq 0$ and $y_i \geq 0$ to maximize

  $$\pi_i = (v_i - y_i) p_i (s_1, s_2, \ldots, s_n) - x_i,$$

  subject to $s_i = f(x_i, y_i)$. 


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Introduction: What is a hybrid contest? (1/2)

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Competition for a government contract or grant:
- **All-pay investments**: Time/effort spent on preparing proposal.
- **Winner-pay investments**: Commit to ambitious customer service.

A political election:
- **All-pay investments**: Campaign expenditures.
- **Winner-pay investments**: Electoral promises (costly if they deviate from the politician’s own ideal policy).

Rent seeking to win monopoly rights of a regulated market:
- **All-pay investments**: Ex ante bribes (how Tullock modeled it).
- **Winner-pay investments**: Conditional bribes.

Tullock’s motivation:
- Empirical studies in the 1950s: DWL appears to be tiny.
- Tullock: Maybe a part of profits adds to the cost of monopoly.
Two earlier papers that model a hybrid contest:

- **Haan and Schonbeek (2003).**
  - They assume Cobb-Douglas—which here is quite restrictive.

- **Melkoyan (2013).**
  - CES but with $\sigma \geq 1$. Symmetric model. Hard to check SOC.
  - My analysis: (i) other approach which yields easy-to-check existence condition; (ii) assumes general production function and CSF; (iii) studies both symmetric and asymmetric models.

Other contest models with more than one influence channel:


- **Multiple all-pay “arms”** (maybe with different costs): Arbatskaya and Mialon (2010).
Literature Review (1/2)

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Multidimensional (procurement) auctions:

- **Che (2003), Branck (1997), Asker and Cantillon (2008).**
  - Firms bid on both price and (many dimensions of) quality.
  - The components of each bid jointly determine a score.
  - Auctioneer chooses bidder with highest score.
- **Differences:**
  - In their models, not both all-pay and winner-pay ingredients.
  - Not a probabilistic CSF.

Optimal design of a research contest: **Che and Gale (2003).**

- A principal wants to procure an innovation.
- Firms choose both quality of innovation and the prize if winning.
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A model of a hybrid contest (1/2)

- $n \geq 2$ contestants try to win an indivisible prize.
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$$\pi_i = (v_i - y_i) p_i(s) - x_i, \quad \text{subject to } s_i = f(x_i, y_i),$$

where $s = (s_1, s_2, \ldots, s_n)$ and $s_i \geq 0$ is contestant $i$’s score.

- $v_i > 0$ is $i$’s valuation of the prize.
- $p_i(s)$ is $i$’s prob. of winning (or contest success function, CSF).
- $x_i$ is the all-pay investment: paid whether $i$ wins or not.
- $y_i$ is the winner-pay investment: paid i.f.f. $i$ wins.

- It is a one-shot game where the contestants choose their investments $(x_i, y_i)$ simultaneously with each other.
Assumptions about $p_i(s)$:
- Twice continuously differentiable in its arguments.
- Strictly increasing and strictly concave in $s_i$.
- Strictly decreasing in $s_j$ for all $j \neq i$.
- The contest is won by someone: $\sum_{j=1}^{n} p_j(s) = 1$.
- Later I assume that $p_i(s)$ is homogeneous in $s$.

Assumptions about $f(x_i, y_i)$:
- Thrice continuously differentiable in its arguments.
- Strictly increasing in each of its arguments.
- Strictly quasiconcave.
- Homogeneous of degree $t > 0$: $\forall k > 0 \ f(kx_i, ky_i) = k^t f(x_i, y_i)$.
- Inada conditions to rule out $x_i = 0$ or $y_i = 0$.

Examples:

$$p_i(s) = \frac{w_i s_i^r}{\sum_{j=1}^{n} w_j s_j^r}, \quad f(x_i, y_i) = \left[ \alpha x^{\frac{\sigma-1}{\sigma}} + (1 - \alpha) y^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$
A model of a hybrid contest (2/2)

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One possible approach:
- Plug the production function into the CSF.
- Take FOCs w.r.t. $x_i$ and $y_i$.
- Used by Haan and Schoonbeek (2003) and Melkoyan (2013), assuming Cobb-Douglas and CES, respectively.

My approach: Solve for contestant $i$’s best reply in two steps:
1. Compute the conditional factor demands.
   - That is, derive optimal $x_i$ and $y_i$, given $s$ (so also given $s_i$).
2. Plug the factor demands into the payoff and then characterize contestant $i$’s optimal score $s_i$ (given $s_{-i}$).

Important advantage: a single choice variable at 2, so easier to determine what conditions are required for equilibrium existence.
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Contestant $i$ solves (for fixed $p_i$): $\min_{x_i, y_i} p_i y_i + x_i$, subject to $f(x_i, y_i) = s_i$.

The first-order conditions ($\lambda_i$ is the Lagrange multiplier):

$$\frac{\partial L_i}{\partial x_i} = 1 - \lambda_i f_1(x_i, y_i) = 0, \quad \frac{\partial L_i}{\partial y_i} = p_i - \lambda_i f_2(x_i, y_i) = 0.$$

So, by combining the FOCs:

$$\frac{1}{p_i} = \frac{f_1(x_i, y_i)}{f_2(x_i, y_i)} \overset{\text{def}}{=} g \left( \frac{x_i}{y_i} \right) \Rightarrow x_i = y_i h \left( \frac{1}{p_i} \right),$$

where $h$ is the inverse of $g$ (i.e., $h \overset{\text{def}}{=} g^{-1}$).

By plugging back into $s_i = f(x_i, y_i)$ and rewriting, we obtain:

$$Y_i(s_i, p_i) = \left[ \frac{s_i}{f(h(1/p_i), 1)} \right]^{1/t}, \quad X_i(s_i, p_i) = Y_i(s_i, p_i) h \left( \frac{1}{p_i} \right).$$

Contestant $i$’s payoff: $\pi_i(s) = p_i(s) v_i - C_i[s_i, p_i(s)]$, where

$$C_i[s_i, p_i(s)] \overset{\text{def}}{=} p_i(s) Y_i[s_i, p_i(s)] + X_i[s_i, p_i(s)].$$

A Nash equilibrium of the hybrid contest:

A profile $s^*$ such that $\pi_i(s^*) \geq \pi_i(s_i, s^*)$, all $i$ and all $s_i \geq 0$. 
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The cost-minimization problem and the $h$ function

(a) Cost minimization.

(b) Graph of the $g$ function.

(c) Graph of the $h$ function.
Equilibrium existence

Define the following elasticities:

- The elasticity of output w.r.t. $x_i$: $\eta \left( \frac{1}{p_i} \right) \equiv \frac{f_1 \left[ h \left( \frac{1}{p_i} \right), 1 \right] h \left( \frac{1}{p_i} \right)}{f \left[ \frac{1}{p_i}, 1 \right]}$.

- The elasticity of substitution: $\sigma \left( \frac{1}{p_i} \right) \equiv -\frac{h' \left( \frac{1}{p_i} \right)}{h \left( \frac{1}{p_i} \right)} \frac{1}{p_i}$.

- The elasticity of the win probability w.r.t. $s_i$: $\varepsilon_i \left( s \right) \equiv \frac{\partial p_i}{\partial s_i} \frac{s_i}{p_i}$.

We have that $\eta \in (0, t)$, $\sigma > 0$, and $\varepsilon_i \in (0, 1)$.

Assumption 1. The production function and the CSF satisfy:

(i) $t \leq 1$ and $\varepsilon_i \left( s \right) \eta \left( \frac{1}{p_i} \right) \sigma \left( \frac{1}{p_i} \right) \leq 2$ (for all $p_i$ and $s$);

Proposition 1. Suppose Assumption 1 is satisfied. Then there exists a pure strategy Nash equilibrium of the hybrid contest.
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  (i) $t \leq 1$ and $\varepsilon_i \left( s \right) \eta \left( \frac{1}{p_i} \right) \sigma \left( \frac{1}{p_i} \right) \leq 2$ (for all $p_i$ and $s$);

- **Proposition 1.** Suppose Assumption 1 is satisfied. Then there exists a pure strategy Nash equilibrium of the hybrid contest.
Equilibrium existence

Define the following elasticities:

- The elasticity of output w.r.t. \( x_i \): 
  \[
  \eta \left( \frac{1}{p_i} \right) \overset{\text{def}}{=} \frac{f_1 \left[ h \left( \frac{1}{p_i} \right), 1 \right] h \left( \frac{1}{p_i} \right)}{f \left[ h \left( \frac{1}{p_i} \right), 1 \right]}. 
  \]

- The elasticity of substitution: 
  \[
  \sigma \left( \frac{1}{p_i} \right) \overset{\text{def}}{=} -\frac{h' \left( \frac{1}{p_i} \right)}{h \left( \frac{1}{p_i} \right)} \frac{1}{p_i}. 
  \]

- The elasticity of the win probability w.r.t. \( s_i \): 
  \[
  \varepsilon_i \left( s \right) \overset{\text{def}}{=} \frac{\partial p_i}{\partial s_i} \frac{s_i}{p_i}. 
  \]

- We have that \( \eta \in (0, t) \), \( \sigma > 0 \), and \( \varepsilon_i \in (0, 1) \).

**Assumption 1.** The production function and the CSF satisfy:

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**Proposition 1.** Suppose Assumption 1 is satisfied. Then there exists a pure strategy Nash equilibrium of the hybrid contest.
Assume a CES production function, \( t = 1, r \leq 1 \), and

\[
p_i(s) = \frac{w_i s^r_i}{\sum_{j=1}^{n} w_j s^r_j} \quad \text{and} \quad p_i(0, \cdots, 0) = \frac{w_i}{\sum_{j=1}^{n} w_j}.
\]

Assumption 1 satisfied

\[
\Theta(\sigma, r) \overset{\text{def}}{=} \frac{\left(\frac{2}{r \sigma - 2}\right)^{\frac{1}{\sigma}}}{1 + \left(\frac{2}{r \sigma - 2}\right)^{\frac{1}{\sigma}}}
\]
To check the SOC with Melkoyan’s analytical approach is cumbersome and in the end he relies on numerical simulations:

\[ \text{[\ldots] one can demonstrate, after a series of tedious algebraic manipulations, that a player’s payoff function is locally concave at the symmetric equilibrium candidate in (7) if and only if [large mathematical expression]. [\ldots] Numerical simulations indicate that this inequality is violated only for extreme values of the parameters [\ldots]. In addition to verifying the local second-order conditions, I have used numerical simulations to verify that the global second-order conditions are satisfied under a wide range of scenarios.} \]
Characterization of equilibrium

- Recall: \( \pi_i(s) = p_i(s)v_i - C_i[s_i, p_i(s)] \).
- The FOC (with an equality if \( s_i > 0 \)):
  \[
  \frac{\partial \pi_i(s)}{\partial s_i} = \frac{\partial p_i(s)}{\partial s_i}v_i - C_1(s_i, p_i) - C_2(s_i, p_i) \frac{\partial p_i(s)}{\partial s_i} \leq 0.
  \]
- Use Shephard’s lemma, \( C_2(s_i, p_i) = Y_i[s_i, p_i(s)] \):
  \[
  [v_i - Y_i(s_i, p_i(s))] \frac{\partial p_i(s)}{\partial s_i} \leq C_1(s_i, p_i), \tag{1}
  \]
  with an equality if \( s_i > 0 \).
- **Proposition 2.** Suppose Assumption 1 is satisfied. Then \( s^* = (s_1^*, \ldots, s_n^*) \) is a pure strategy Nash equilibrium of the hybrid contest if and only if condition (1) holds, with equality if \( s_i^* > 0 \), for each contestant \( i \).
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Assumption 2. The CSF is symmetric and homogeneous of degree 0.

Note that, thanks to Assumption 2:

\[
\frac{\partial p_i(s, s, \ldots, s)}{\partial s_i} = \frac{\widehat{\varepsilon}(n)}{ns}, \text{ where } \widehat{\varepsilon}(n) \overset{\text{def}}{=} \varepsilon_i (1, 1, \ldots, 1).
\]

Use this in the FOC and impose symmetry:

\[
(v - y^*) \frac{\widehat{\varepsilon}(n)}{ns^*} = C_1 \begin{bmatrix} s^*, \frac{1}{n} \end{bmatrix} = \frac{1}{ts^*} C \begin{bmatrix} s^*, \frac{1}{n} \end{bmatrix} = \frac{1}{ts^*} \begin{bmatrix} y^* + x^* \\ n \end{bmatrix}
\]

\[\Leftrightarrow (v - y^*) t\widehat{\varepsilon}(n) = y^* + nx^* . \text{ And from before, } x^* = h(n)y^* .
\]

The last equalities are linear in \(x^*\) and \(y^*\), so easy to solve.

Proposition 3. Within the family of sym. eq., there is a unique pure strategy equilibrium: \(s^* = f[h(n), 1] (y^*)^t, x^* = h(n)y^* , \text{ and } y^* = \frac{t\widehat{\varepsilon}(n)v}{1 + nh(n) + t\widehat{\varepsilon}(n)}\).
A Symmetric Hybrid Contest (1/4)

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  \]
  
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  \[
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  \begin{bmatrix}
  s^* \\
  \frac{1}{n}
  \end{bmatrix} = \frac{1}{ts^*} C 
  \begin{bmatrix}
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  \frac{1}{n}
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  \]
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A Symmetric Hybrid Contest (1/4)

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**Assumption 2.** The CSF is symmetric and homogeneous of degree 0.

Note that, thanks to Assumption 2:

\[
\frac{\partial p_i(s, s, \ldots, s)}{\partial s_i} = \hat{\varepsilon}(n) \frac{n}{ns}, \quad \text{where} \quad \hat{\varepsilon}(n) \overset{\text{def}}{=} \varepsilon_i(1, 1, \ldots, 1).
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Use this in the FOC and impose symmetry:

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**A Symmetric Hybrid Contest (1/4)**

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\]
Proposition 4. Effect of more contestants on $x^*$ and $y^*$:

\[
\frac{\partial x^*}{\partial n} < 0 \iff \sigma(n) > \frac{n(n - 2)h(n) - 1}{(n - 1)[1 + t\widehat{\epsilon}(n)]},
\]
\[
\frac{\partial y^*}{\partial n} > 0 \iff \sigma(n) > \frac{n(n - 2)h(n) - 1}{(n - 1)nh(n)};
\]

and if $\sigma(n) \geq 1$, then necessarily $\frac{\partial x^*}{\partial n} < 0$ and $\frac{\partial y^*}{\partial n} > 0$.

In order to understand the above:

- More contestants means a lower probability of winning.
- This lowers the relative cost of investing in $y_i$.
- So whenever $\sigma(n)$ is sufficiently large, $\frac{\partial y^*}{\partial n} > 0$ and $\frac{\partial x^*}{\partial n} < 0$.
- But if $\sigma(n)$ small, the derivatives must have the same sign. For:

\[
\frac{\partial y^*}{\partial n} \frac{n}{y^*} = \sigma(n) + \frac{\partial x^*}{\partial n} \frac{n}{x^*} \quad \text{(follows from } x^* = h(n)y^* \text{)}.
\]

As $\sigma(n) \to 0$, the production function requires $x_i$ and $y_i$ to be used in fixed proportions (a Leontief production technology).
The total amount of equilibrium expenditures in the symmetric hybrid model is defined as $R^H \overset{\text{def}}{=} nC \left[ s^*, \frac{1}{n} \right]$. The corresponding amount in the all-pay contest: $R^A = t\widehat{e}(n)v$.

**Proposition 5, part (a).** In the symmetric model:

$$R^H = (1 - \frac{y^*}{v})R^A = \left[ \frac{1}{v [1 + nh(n)]} + \frac{1}{R^A} \right]^{-1}.$$

In particular, for any finite $n$, we have $R^H < R^A$.

The payoff suggests the intuition: $\pi_i = (v_i - y_i)p_i(s) - x_i$.

**Proposition 5, part (b).** In the symmetric model, suppose $p_i(s) = \phi(s_i)/\sum_{j=1}^n \phi(s_j)$, where $\phi$ is a strictly increasing and concave function satisfying $\phi(0) = 0$.

Then $R^H$ is weakly increasing in $n$ if and only if: (i)

$$\sigma(n) \leq 1 + \frac{4n}{tr(n - 1)^2};$$

or (ii) inequality (2) is violated and $h(n) \notin (\Xi_L, \Xi_H)$. See figure!
The total amount of equilibrium expenditures in the symmetric hybrid model is defined as $R^H \overset{\text{def}}{=} nC \left[ s^*, \frac{1}{n} \right]$. The corresponding amount in the all-pay contest: $R^A = t\hat{e}(n)v$.

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Assume CES, $t = 1$, and $n = 10$. 

Assumption 1 satisfied

$R^H$ decreasing in $n$ at $n = 10$
I assume $n = 2$ and I study three models:
- The CSF is biased in favor of one contestant.
- One contestant has a higher valuation than the other.
- I also endogenize the degree of bias.

**Assumption 3.** The CSF is given by

$$p_i(s) = \frac{w_i s_i^r}{w_1 s_1^r + w_2 s_2^r}.$$ 

The following three equations define equilibrium values of $p_1^*$, $y_1^*$, and $y_2^*$:

$$y_i^* = \frac{rtp_i^*(1 - p_i^*) v_i}{rtp_i^*(1 - p_i^*) + p_i^* + h \left( \frac{1}{p_i^*} \right)}, \quad \text{for } i = 1, 2, \text{ and } \Upsilon(p_1^*) = 0,$$

where

$$\Upsilon(p_1) \overset{\text{def}}{=} \frac{w_2 v_2^r}{w_1 v_1^r} p_1 f \left[ h \left( \frac{1}{1 - p_1} \right), 1 \right]^r \left[ rtp_1(1 - p_1) + 1 - p_1 + h \left( \frac{1}{1 - p_1} \right) \right]^r - \left( 1 - p_1 \right) f \left[ h \left( \frac{1}{p_1} \right), 1 \right]^r \left[ rtp_1(1 - p_1) + p_1 + h \left( \frac{1}{p_1} \right) \right]^r.$$

The equilibrium is unique if $r \eta \left( \frac{1}{p_i} \right) \sigma \left( \frac{1}{p_i} \right) \leq 1.$
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$$y_i^* = \frac{rtp_i^*(1 - p_i^*) v_i}{rt p_i^*(1 - p_i^*) + p_i^* + h\left(\frac{1}{p_i^*}\right)}, \quad \text{for } i = 1, 2, \text{ and } \gamma(p_1^*) = 0,$$ 

where

$$\gamma(p_1) \overset{\text{def}}{=} \frac{w_2 v_2^r}{w_1 v_1^r} p_1 f \left[h\left(\frac{1}{1-p_1}\right), 1\right]^r - \frac{(1 - p_1) f \left[h\left(\frac{1}{p_1}\right), 1\right]^r}{r t p_1 (1 - p_1) + p_1 + h\left(\frac{1}{p_1}\right)}.$$ 

The equilibrium is unique if $r \eta \left(\frac{1}{p_i}\right) \sigma \left(\frac{1}{p_i}\right) \leq 1.$
I assume $n = 2$ and I study three models:

- The CSF is biased in favor of one contestant.
- One contestant has a higher valuation than the other.
- I also endogenize the degree of bias.

**Assumption 3.** The CSF is given by

$$p_i(s) = \frac{w_i s_i^r}{w_1 s_1^r + w_2 s_2^r}.$$

The following three equations define equilibrium values of $p_1^*, y_1^*$, and $y_2^*$:

$$y_i^* = \frac{rtp_i^*(1 - p_i^*)v_i}{rtp_i^*(1 - p_i^*) + p_i^* + h\left(\frac{1}{p_i^*}\right)}, \quad \text{for } i = 1, 2, \text{ and } \Upsilon(p_1^*) = 0,$$

where

$$\Upsilon(p_1) \overset{\text{def}}{=} \frac{w_2 v_2^r}{w_1 v_1^r} p_1 f \left[ h\left(\frac{1}{1-p_1}\right), 1 \right]^r - \frac{(1 - p_1) f \left[ h\left(\frac{1}{p_1}\right), 1 \right]^r}{rtp_1(1 - p_1) + p_1 + h\left(\frac{1}{p_1}\right)}.$$

The equilibrium is unique if $r\eta\left(\frac{1}{p_i}\right) \sigma\left(\frac{1}{p_i}\right) \leq 1$. 

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A Biased decision process \((w_1 \neq w_2 \text{ but } \nu_1 = \nu_2)\)

Among the results:

(a) \(p_1^* > p_2^* \iff y_1^* < y_2^* \iff C(s_1^*, p_1^*) > C(s_2^*, p_2^*)\).

(b) Evaluated at symmetry \((w_1 = w_2)\): \(\frac{\partial p_1^*}{\partial w_1} > 0\),

\[
\frac{\partial y_1^*}{\partial w_1} < 0, \quad \frac{\partial y_2^*}{\partial w_1} > 0, \quad \frac{\partial x_1^*}{\partial w_1} > 0 \iff \frac{\partial x_2^*}{\partial w_1} < 0 \iff \sigma(2) > \frac{2}{2 + rt}.
\]

Different valuations \((\nu_1 \neq \nu_2 \text{ but } w_1 = w_2)\)

Among the results:

(a) \(p_1^* > p_2^* \iff \frac{y_1^*}{\nu_1} < \frac{y_2^*}{\nu_2}\).

(b) \(\nu_1 - y_1^* > \nu_2 - y_2^* \iff C(s_1^*, p_1^*) > C(s_2^*, p_2^*)\).
Asymmetric Hybrid Contests (2/4)

A Biased decision process \((w_1 \neq w_2 \text{ but } v_1 = v_2)\)

- Among the results:
  
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Different valuations \((v_1 \neq v_2 \text{ but } w_1 = w_2)\)

- Among the results:
  
  (a) \(p_1^* > p_2^* \iff \frac{y_1^*}{v_1} < \frac{y_2^*}{v_2}.\)

  (b) \(v_1 - y_1^* > v_2 - y_2^* \iff C(s_1^*, p_1^*) > C(s_2^*, p_2^*).\)
An Endogenous Bias ($w_1$ chosen, but $v_1 \geq v_2$ and $w_2$ fixed)

- Timing of events in the game:
  1. A principal chooses $w_1$ to maximize $R^H = C(s_1^*, p_1^*) + C(s_2^*, p_2^*)$.
  2. $w_1$ becomes common knowledge and the contestants interact as in the previous analysis.

- Assumption 3. The production function is of Cobb-Douglas form: $f(x_i, y_i) = x_i^\alpha y_i^\beta$, for $\alpha > 0$ and $\beta > 0$.

- Results: The equilibrium values of $p_1$ and $w_1$ satisfy:
  - If $v_1 = v_2$, then $\hat{p}_1 = \frac{1}{2}$ and $\hat{w}_1 = w_2$.
  - If $v_1 > v_2$, then $\hat{p}_1 > \frac{1}{2}$.
  - If $v_1 > v_2$, then $\hat{w}_1 < w_2$ at least if $|v_1 - v_2|$ is very small or big.

- My intuition for results:
  - Contestant 1 is more valuable as a contributor (as $v_1 > v_2$).
  - Hence, she should be encouraged to use $x_1$, as all-pay investments are more conducive to large expenditures.
  - This is achieved by making winner-pay inv. costly: $\hat{p}_1 > \frac{1}{2}$.
  - To generate $\hat{p}_1 > \frac{1}{2}$, $v_1 > v_2$ is more than enough, so bias can be in favor of Contestant 2.
  - Might not be robust.
An Endogenous Bias ($w_1$ chosen, but $v_1 \geq v_2$ and $w_2$ fixed)

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- Might not be robust.
Numerical example \( t = r = v_2 = w_2 = 1 \)

- Plot of \( \hat{p}_1 \) and \( \hat{w}_1 \) against \( v_1 \) for three different values of \( \alpha \): 0.9 (the blue, dotted curve), 0.5 (the green, dashed curve), and 0.1 (the red, solid curve).

(a) The high-valuation contestant’s probability of winning.

(b) The weight in the CSF that is assigned to the high-valuation contestant’s score.
Main results and contributions: (1/1)

1. The analytical approach (borrowing from producer theory):
   - Generality, tractability, and an existence condition.

2. A larger $n$ leads to substitution away from all-pay investments.
   - But only if the elasticity of substitution is large enough.

3. Total expenditures always lower in hybrid contest than in all-pay.

4. Total exp’tures can be decreasing in $n$ (also shown by Melkoyan).

5. Asym. contests (in terms of valuations and bias): Sharp predictions about relative size of investm’s and of expenditures.

6. Endogenous bias: High-valuation contestant more likely to win but the bias is against her (the latter might not be robust).
1. Sequential moves: first \((x_1, y_1)\), then \((x_2, y_2)\).

2. Applying the producer theory approach to other contest models with multiple influence channels.

3. Experimental testing.
   - Relatively sharp predictions.
   - But risk neutrality might be an issue?

4. Further work on asymmetric contests.
   - More than two contestants.
   - Can a contestant be hurt by a bias in favor of her?
   - Can a contestant benefit from an increase in rival’s valuation?

5. Contest design in broader settings.