# Hybrid All-Pay and Winner-Pay Contests

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**Abstract**

In many contests in economic and political life, both all-pay and winner-pay expenditures matter for winning. This paper studies such hybrid contests under symmetry and asymmetry. The symmetric model assumes very little structure but yields a simple closed-form solution. More contestants tend to lead to substitution toward winner-pay investments, and total expenditures are always lower than in the corresponding all-pay contest. With a biased decision process and two contestants, the favored contestant wins with a higher likelihood, chooses less winner-pay investments, and contributes more to total expenditures. An endogenous bias that maximizes total expenditures disfavors the high-valuation contestant but still makes her the more likely one to win.

**Keywords**: rent-seeking, lobbying, influence activities, multiple influence channels, producer theory

**JEL classification**: C72, D24, D72, D74,

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As defined by the dictionary, a contest is "a struggle for superiority or victory between rivals" (Soukhanov, 1992). Situations that involve such contests are commonplace in economic and political life. Examples include marketing, advertising, litigation, relative reward schemes in firms, beauty contests between firms, rent-seeking for rents allocated by a public regulator, political competition, patent races, sports, military combat, and war.[1] Indeed, there exists a vast theoretical literature that studies contests by modeling them as a non-cooperative game.[2] A common approach is to assume that each one of a number of contestants chooses an effort level. Through a postulated contest success function (CSF), the effort levels jointly determine the probability that a given contestant wins the contest. The winner is awarded a prize. Within this framework, scholars have studied questions about, for example, how much effort an individual contestant exerts, how the sum of effort costs relates to the value of the prize (i.e., the dissipation rate), and how individual and total effort costs are affected by an increase in the number of contestants and by changes in the design of the contest (e.g., the timing of the game or alternative prize structures).

One feature of the above, standard, framework is that each contestant's effort is modeled as an *all-pay investment*: The investment cost is incurred regardless of whether the contestant wins or not. For example, in the competitive bidding to host the Olympic games, candidate cities spend money upfront, with the goal of bribing or otherwise persuading members of the International Olympic Committee; in case a city is not awarded the Games, the money is forfeited. Alternatively, we could think of each contestant's effort as a *winner-pay investment*, meaning that it is contingent on actually winning the contest. For example, a candidate city may commit to build new stadia and other infrastructure and to invest in ambitious safety arrangements if being awarded the Games; or the candidate city offers bribes that are contingent on winning.

In many situations, including the bidding for the Olympics, the contestants can arguably make *both* all-pay and winner-pay investments. Moreover, the extent to which they choose to use each one of these instruments to exert influence is likely to depend on the contest technology and the nature of the strategic environment in which the contestants interact. Other examples include (i) the competition for a government contract or a grant and (ii) a political election.[3] In a contest for a government contract or a grant, the contestants can, on the one hand, spend time and effort preparing their proposal and, on the other hand, commit to actions to be taken if being awarded the contract/grant (like providing ambitious and costly customer service). In a political contest, a candidate can increase her chances of being elected both by making campaign expenditures and by making electoral promises (the latter is costly if the promises deviate from the candidate's ideal policy). While the campaign expenditures are paid upfront, the cost of fulfilling campaign promises are incurred only if the candidate wins the election.

In this paper, I develop a framework for hybrid contests where contestants can make both all-pay and winner-pay investment. I then use this framework to study, both in symmetric and asymmetric environments, the incentives of contestants to invest in each of the two influence

[1] This list is borrowed from Konrad (2009, p. 1).

[2] For surveys of this literature, see Nitzan (1994), Konrad (2009), and Corchón and Serena (2018). For a recent survey of the experimental literature, including a useful introduction to some important modeling approaches, see Dechenaux, Kovenock, and Sheremeta (2015).

[3] These example are discussed also by Melkonyan (2013).

channels; how the contestants optimally mix between all-pay and winner-pay investments; and how the equilibrium investment levels and the dissipation rate depend on the number of contestants, the contest technology, and other aspects of the environment. Finally, I ask what bias in the CSF should be chosen if the contestants have different valuations and the objective is to maximize total equilibrium expenditures.

I set up the formal model in Section 2. In this model there are $n$ contestants who, simultaneously with each other, commit to an all-pay investment level and a winner-pay investment level. These investments jointly generate each contestant's *score*, according to a production function. The scores of the $n$ contestants then, through a CSF, determine each contestant's probability of winning. The economically important assumptions that I make about the score production function is that it is homogeneous and strictly quasiconcave. The CSF is assumed to be strictly concave in the own score (in the analysis in Sections 4 and 5, it is also assumed to be homogeneous).

In Section 3, I first provide sufficient conditions for existence of a pure strategy equilibrium of the hybrid contest (Proposition 1). These conditions require that that the returns to scale associated with the score production function are not too strong. Moreover, for equilibrium existence to be guaranteed, it helps if the elasticity of substitution between the two kinds of investment is not too large; however, in an example with a constant elasticity of substitution (CES), I show that an equilibrium exists also for arbitrarily large values of that elasticity, provided that winner-pay investments are sufficiently important in the score production function. I further characterize the contestants' equilibrium behavior (Proposition 2). In Section 4, I then study a symmetric version of the model, where the CSF is assumed to be homogeneous. In spite of the fact that both the CSF and the production function are general, the model gives rise to a closed-form solution and this solution is quite simple (Proposition 3). The solution is stated partly in terms of a function $h$, which is defined as the inverse of the marginal rate of technical substitution between the two kinds of investment. In a symmetric equilibrium, the argument of $h$ is the number of contestants, $n$.

The comparative statics analysis for the symmetric model shows, among other things, that if the score production function is such that it is relatively easy to substitute between the two kinds of investment, then, as the number of contestants ($n$) increases, each contestant's winner-pay investment goes up and her all-pay investment goes down. The reason is that a larger $n$ implies a lower probability of winning, which effectively lowers the relative cost of winner-pay investments. However, if it is sufficiently difficult to substitute between the two kinds of investment, then the winner-pay and the all-pay investment levels move in the same direction—which direction depends on parameter values—as $n$ goes up (Proposition 4). Section 4 also studies the total amount of expenditures in the symmetric model. It turns out that, for any finite number of contestants, the hybrid contest always gives rise to a strictly smaller amount of total expenditures than the corresponding all-pay contest (Proposition 5). The reason is that, in a hybrid contest, winning the prize is worth less—namely, the gross valuation minus the winner-pay investment. This creates a shift in a contestant's best reply function: For any given behavior of the rivals, she has an incentive to choose lower investment levels. This is true for all contestants, and the result is an equilibrium with lower investment levels and expenditures.

The result that the hybrid contest yields a strictly smaller amount of total expenditures holds also for an infinitely large number of contestants, as long as the limit of $nh(n)$ as $n \to \infty$ is finite; if that limit is infinite, then the limit value of the total expenditures is the same in the two models (Proposition 6). For a CES production function, the limit of $nh(n)$ as $n \to \infty$ is finite if and only if $\sigma \geq 1$, where $\sigma$ is the elasticity of substitution. Intuitively, winner-pay investments are less conducive to large expenditures than all-pay investments are; moreover, for $\sigma \geq 1$ it is relatively easy for the contestants to substitute away from all-pay investments to winner-pay investments when the number of contestants goes up.

In Section 5, I study three asymmetric versions of the model, all with two contestants. I first formulate a framework that encompasses all three models and prove a characterization result as well as a sufficient condition for equilibrium uniqueness (Proposition 7). After that I turn to the first one of the three more specific models: a contest in which the CSF is biased in favor of one of the contestants. At an equilibrium of this contest, the contestant who wins with the higher likelihood also (i) chooses a smaller winner-pay investment and (ii) contributes more to the expected total amount of expenditures. Under the assumption that the asymmetry is small, I show that the contestant who wins with the higher likelihood must be the one who is favored by the CSF. What is the effect on the investment levels of an increase in the bias? There are, depending on how easy it is to substitute, two possibilities. If the elasticity of substitution is relatively high, then the favored contestant does less of winner-pay and more of all-pay investment, while her rival does the opposite; but if the elasticity of substitution is low enough, then the favored contestant does less of both kinds of investment and her rival does more of both of them (Proposition 8).

In the second asymmetric contest the contestants are assumed to have different valuations for winning the prize. Among the results is that (for a small asymmetry) the contestant with the higher valuation wins with the highest likelihood. In contrast to the model with a biased decision process, here the contestant who wins with the higher likelihood does not necessarily choose a smaller amount of winner-pay investments—this happens only when it is sufficiently easy to substitute (Proposition 9).

In the third asymmetric contest there is both a possible bias in the CSF and different valuations. Moreover, the bias (if any) is assumed to be chosen by a principal who wants to maximize the expected total equilibrium expenditures. The final result of the paper states that the optimal bias disfavors the high-valuation contestant but still makes her win with the highest probability (Proposition 10). The reason why the high-valuation contestant is made to win with the highest probability is that she is the more valuable contributor to the overall expenditures. Thus, this contestant should be encouraged to use all-pay investments, as these are conducive to high expenditures. This can be achieved by making her win probability high (for then all-pay investments are relatively inexpensive).

# 1 Related Literature

Haan and Schoonbeek (2003) and Melkonyan (2013) study special cases of the present framework. The former paper assumes a Cobb-Douglas production function (in addition, the exponents in this function both equal unity) and a lottery CSF. It also derives results for an asymmetric contest where the contestants differ from each other with respect to their valuations. Melkonyan assumes a CES production function, but with the restriction that the elasticity of substitution cannot be below unity; his CSF is of the Tullock (1980) form. Moreover, he studies only a symmetric contest. The present analysis, in contrast, assumes a general production function and a general CSF (the essential assumptions are, for the former, strict quasi-concavity and homogeneity and, for the latter, strict concavity in the own score and homogeneity). In the symmetric version of the model, these more general assumptions still allow for a closed-form solution, which is quite simple. In addition, the general analysis is actually simpler and more tractable than the analysis of the models using a specific functional form. The more general analysis is possible thanks to an alternative methodology. Instead of plugging in the score production function into the CSF and then take two first-order conditions for each contestant, the idea is to derive a contestant's best reply in two steps.[4] First, I fix a contestant's score and solve for the optimal levels of all-pay and winner-pay investments that can produce that score. In producer theory language, I compute two conditional factor demand functions by solving a cost-minimization problem. Second, using the conditional factor demand functions I can easily derive a contestant's optimal score and thus also her best reply. One reason why this approach is helpful is that, at the second step, each contestant has a single choice variable, which makes it much easier to determine what conditions are required for equilibrium existence. Indeed, an important contribution relative to Melkonyan's (2013) analysis is to formulate a simple sufficient condition for equilibrium existence, stated in terms of some key elasticities.[5]

Siegel (2010) formulates an interesting and quite general framework that accommodates both all-pay and winner-pay (or, using his terminology, conditional and unconditional) investments. However, each contestant's investment is one-dimensional. The single investment level leads, according to an exogenous rule, to costs that are incurred partly conditional, partly unconditional, on winning. For example, in a special case of his model, a constant fraction of the cost is paid only if winning and the remaining fraction is always paid. This model feature means that there cannot be any substitution from, say, all-pay investments to winner-pay investments when the economic environment changes, which is an important aspect of the hybrid contest. Another important model feature that distinguishes Siegel's framework from the one in the present paper is that, in his setting, a contestant who makes a strictly greater effort than all her rivals always wins for sure, like in an all-pay auction: The CSF involves no uncertainty (except possibly when there are ties).

Also related to the present analysis are papers that model contests with more than one influence channel (or multi-dimensional efforts), although not in the form of all-pay and winner-

[4]The first step of this approach has earlier been employed, in a contest model context, in an example discussed by Kaplan and Wettstein (2006, p. 1352 and in particular fn. 1).

[5]To check the second-order conditions in Melkonyan's framework, using his analytical approach, is cumbersome, and when Melkonyan does it he partly relies on numerical simulations. To get a sense of how cumbersome it is, consider the following passage from Melkonyan (2013, p. 976): "[...] one can demonstrate, after a series of tedious algebraic manipulations, that a player's payoff function is locally concave at the symmetric equilibrium candidate in (7) if and only if [large mathematical expression]. One can verify that the left-hand side of the above inequality is neither positive for all parameter values nor negative. An examination of this expression also reveals that the set of parameter values for which the determinant of the Hessian matrix is positive has a strictly positive measure. Numerical simulations indicate that this inequality is violated only for 'extreme values' of the parameters [...]. In addition to verifying the local second-order conditions, I have used numerical simulations to verify that the global second-order conditions are satisfied under a wide range of scenarios."

pay investments. These papers can be grouped into (at least) three categories. First, there is a literature on sabotage in contests, where contestants exert effort both to improve the own performance and to sabotage the rivals' performances. See, e.g., Konrad (2000), who uses a Tullock CSF, and Chen (2003), who uses a rank-order tournament à la Lazear and Rosen (1981). Second, some works study contest models of war and conflict where the contestants allocate their endowments between two activities: production and appropriation. Early contributions to this strand of literature are Hirshleifer (1991) and Skaperdas and Syropoulos (1997). Third, a number of papers extend the standard all-pay contest by allowing for two or more "arms" of the influence activities, although all arms are of the all-pay nature. A recent example of this is Arbatskaya and Mialon (2010), who assume that each contestant chooses a whole vector of all-pay effort levels and that the linear effort costs may differ across arms and contestants. The contestants' effort levels jointly determine the win probabilities thorough a Tullock CSF where the effects of the different arms are aggregated by a Cobb-Douglas function. Arbatskaya and Mialon also, within their setting, provide an axiomatic justification for this Tullock-Cobb-Douglas functional form. Other contributions within this third category include Clark and Konrad (2007), who study a two-player Tullock contest with multi-dimensional efforts and where a contestant must win in a certain number of these dimensions in order to be awarded the prize.[6]

A few papers have studied models in which the contestants can make *only* winner-pay investments. Yates (2011) formulates and solves a fairly general such model with two contestants. He considers both a symmetric and an asymmetric setting and he also presents some results for an example with private information about each contestant's valuation. Wärneryd (2000) models a court case in which two parties can either represent themselves or hire lawyers. In the latter case, each contestant needs to pay a lawyer's fee only if winning the case; this part of the game is thus modeled as a winner-pay contest. The main point of Wärneryd's paper is that both parties prefer compulsory representation by lawyers, as this helps to reduce expenditures. This finding is related to the result in the present paper that the hybrid contest (and thus also a pure winner-pay contest) give rise to less total expenditures than the all-pay contest. Matros and Armanios (2009) study an $n$-player all-pay Tullock contest with reimbursements. That is, the authors assume that after a win or a loss, respectively, a certain exogenous fraction of the expenditures that have been paid upfront are reimbursed to the contestant. For particular parameter values, this contest simplifies to a winner-pay contest (and for other particular parameter values, it amounts to special case of Siegel (2010), discussed above).

There is also a related literature on multidimensional auctions (Che, 1993; Branco, 1997; Asker and Cantillon, 2008). In the models in this literature, firms bid on both price and (possibly many dimensions of) quality. Then the components of each bid jointly determine a score, according to a previously announced scoring rule. The buyer chooses the bid with the highest score (and, depending on the auction rules, the winning firm supplies either its own submitted score or that of the second-highest scorer). In some of the papers, the scoring rule is endogenous. Che (1993) studies an auction with a one-dimensional quality and a quasi-linear scoring rule; Branco (1997) extends Che's analysis to the case with correlated costs (but independent signals); and Asker and Cantillon (2008) add, among other things, multidimensional private

[6] Another literature that can be said to belong to this third category is the one on conflicts with multiple "battlefields," surveyed by Kovenock and Roberson (2012).

information to Che's analysis. Apart from the common feature of a score production function, an interesting parallel with my analysis of the hybrid contest is that Che (1993) solves his model using a two-step procedure: A supplier first determines the optimal quality-price mix for any given score level, which reduces the two-dimensional problem to a single-dimensional one. However, there are also important differences. For example, in the multidimensional auction, the selection of a winner is made using the auction rule, as opposed to a probabilistic CSF. Furthermore, in the hybrid contest, the two dimensions of the bid refer to an all-pay investment and a winner-pay investment. In the multidimensional auction, in contrast, all costs are incurred and the price is paid conditional on winning, so only winner-pay investments are possible.

Finally, there is an interesting paper by Che and Gale (2003) that studies the optimal design of a research contest. In the contest that they study, a buyer wants to procure an innovation. Each one of a number of firms can submit a bid that specifies both the quality of the firm's innovation and the prize it is awarded if it wins the contest. The winner is the firm whose bid yields the highest net value (level of quality minus the required prize) for the buyer. Choosing a strictly positive quality requires an ex ante investment. Hence, the firm's investment cost plays a similar role to the all-pay investment in the hybrid contest. In contrast, the prize that the firm specifies is awarded only if the firm win's the contest, which makes the negative of the prize tantamount to the winner-pay investment in the hybrid contest. There are also, however, several differences between the setups. The effective CSF in Che and Gale's model is perfectly discriminatory, which implies that the firms use mixed investment strategies in equilibrium. Moreover, the effective score production function is linear and, as a consequence, substitution between prize and quality does not play a role. More generally, Che and Gale use a mechanism-design approach and derive the optimal mechanism in an environment in which prizes but not investments are enforceable by courts.

## 2 A Model of a Hybrid Contest

Consider the following model of a *hybrid contest*, that is, a contest in which the outcome is determined by both all-pay and winner-pay investments. There are $n \geq 2$ economic agents, or contestants, who try to win an indivisible prize. Contestant $i$'s valuation of the prize equals $v_i > 0$ and her probability of winning is determined by the contest success function (CSF)

$$p_i(\mathbf{s}) \in [0,1], \qquad \text{with} \quad \sum_{i=1}^{n} p_i(\mathbf{0}) \leq 1 \qquad \text{and} \qquad \sum_{i=1}^{n} p_i(\mathbf{s}) = 1 \text{ for all } \mathbf{s} \neq \mathbf{0}, \tag{1}$$

where $\mathbf{0}$ is the $n$-dimensional zero vector, $\mathbf{s} = (s_1, s_2, \ldots, s_n)$, and $s_i \geq 0$ is contestant $i$'s *score*. The function $p_i$ is twice continuously differentiable for all $\mathbf{s} \in \Re^n_+ \setminus \{\mathbf{0}\}$.[7] Moreover, it is strictly increasing and strictly concave in $s_i$, and it is strictly decreasing in $s_j$ for all $j \neq i$. In addition, if $s_i = 0$ and $s_j > 0$ for some $j \neq i$, then $p_i(\mathbf{s}) = 0$. Any values of $p_i(\mathbf{0})$ that are consistent with (1) are allowed, although it is assumed that $p_i(\mathbf{0}) < 1$ for all $i$.

[7]Note that the continuity assumption rules out a perfectly discriminatory CSF.

| Name | $f(x,y)$ | $h(m)$ | $\sigma(m)$ | $\eta(m)$ |
|---|---|---|---|---|
| Leontief | $\min\left\{\frac{x}{\alpha},\frac{y}{\beta}\right\}$ | n.a. | 0 | n.a. |
| Linear technology | $\alpha x+\beta y$ | n.a. | $\infty$ | n.a. |
| Cobb-Douglas | $x^{\alpha}y^{\beta}$ | $\frac{\alpha}{\beta m}$ | 1 | $\frac{\alpha}{\alpha+\beta}$ |
| CES | $\left[\alpha x^{\frac{\sigma-1}{\sigma}}+(1-\alpha)y^{\frac{\sigma-1}{\sigma}}\right]^{\frac{t\sigma}{\sigma-1}}$ | $\left[\frac{\alpha}{(1-\alpha)m}\right]^{\sigma}$ | $\sigma$ | $\frac{t\alpha^{\sigma}m^{1-\sigma}}{\alpha^{\sigma}m^{1-\sigma}+(1-\alpha)^{\sigma}}$ |

**Table 1:** Examples of production functions. More examples and relevant references can be found in Nadiri (1982).

Contestant $i$'s score $s_i$ is determined by the production function $s_i = f(x_i, y_i)$. The variables $x_i \geq 0$ and $y_i \geq 0$ are both chosen by contestant $i$. The first one, $x_i$, is the all-pay investment; this is the amount of money the contestant pays regardless of whether she wins the prize or not. The second variable, $y_i$, is the winner-pay investment: the amount contestant $i$ pays if and only if she wins the prize. The production function $f(x_i, y_i)$ is strictly quasiconcave, three times continuously differentiable, and strictly increasing in each of its arguments. Moreover, the function satisfies $f(0,0) = 0$ and the following Inada conditions:[8] $\lim_{x_i \to 0} f_1(x_i, y_i) = \infty$ for all $y_i > 0$, and $\lim_{y_i \to 0} f_2(x_i, y_i) = \infty$ for all $x_i > 0$. Finally, it is homogeneous of degree $t > 0$; formally, for all $k > 0$, $f(kx_i, ky_i) = k^t f(x_i, y_i)$. Some of these assumptions rule out a linear production technology and a Leontief production technology. However, the analysis below can deal with those technologies as limit cases.

The contestants are risk neutral, which means that contestant $i$ maximizes the following expected payoff:

$$\pi_i = (v_i - y_i)\, p_i(\mathbf{s}) - x_i, \tag{2}$$

subject to $s_i = f(x_i, y_i)$. The contestants choose their investments $(x_i, y_i)$ simultaneously with each other and they interact only once.

## 3 Existence and Characterization of Equilibrium

I will confine attention to pure strategy Nash equilibria of the game. In order to characterize these equilibria, one possible approach would be to simply plug the constraint $s_i = f(x_i, y_i)$ into the payoff function (2) and then, for each contestant, derive one first-order conditions for each of the two choice variables. However, that methodology makes it hard to determine under what circumstances a pure strategy equilibrium exists (which is a real issue in this model). It also makes the algebra cumbersome, which is a problem in itself and also makes it difficult to detect the underlying economic logic of the model. I will instead use an alternative approach that makes it easier to identify a sufficient condition for equilibrium existence. In addition, this approach makes the analysis significantly more tractable, in spite of the fact that relatively little structure is imposed on the model.

Contestant $i$'s best reply is defined, in the usual way, as her optimal choice of $x_i$ and $y_i$,

[8] The subscript 1 (2, resp.) denotes the partial derivative of $f$ with respect to the first (second, resp.) argument.

given some particular actions of the other contestants, $\mathbf{s_{-i}} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$. The idea behind the approach that I will employ is to derive contestant $i$'s best reply in two steps:

1. First I derive the optimal $x_i$ and $y_i$, given some value of $\mathbf{s}$ (so, in particular, given the own score $s_i$). In producer theory language, I compute the conditional factor demand functions by solving a cost-minimization problem.

2. With the conditional factor demand functions at hand I can then, at the second step, characterize contestant $i$'s optimal score $s_i$ (given $\mathbf{s_{-i}}$), which in turn yields the optimal values of $x_i$ and $y_i$ (given $\mathbf{s_{-i}}$).

One reason why this approach is helpful is that, at the second step, each contestant has a single choice variable, which makes it much easier to determine what conditions are required for equilibrium existence.

### 3.1 Step 1: The Cost-Minimization Problem

At step 1 the contestant treats the probability of winning, $p_i$, as a parameter and chooses $x_i$ and $y_i$ so as to minimize the expected costs $p_i y_i + x_i$, subject to the constraint $f(x_i, y_i) = s_i$.[9] (Thanks to the Inada conditions stated in the model description, the constraints $x_i \geq 0$ and $y_i \geq 0$ do not bind and we can thus disregard them.) This is equivalent to a standard cost-minimization problem for a price-taking firm, as studied in microeconomics textbooks (see, e.g., Mas-Colell et al., 1995, Ch. 5), except that here the "prices" of input $x_i$ and $y_i$ equal unity and $p_i$, respectively. The problem is depicted in panel (a) of Figure 1.

The Lagrangian of the cost-minimization problem can be written as $\mathcal{L}_i = p_i y_i + x_i - \lambda_i [f(x_i, y_i) - s_i]$, where $\lambda_i$ is the shadow price associated with the constraint and where the argument of $p_i$ has been suppressed. The necessary first-order conditions are:

$$\frac{\partial \mathcal{L}_i}{\partial x_i} = 1 - \lambda_i f_1(x_i, y_i) = 0, \qquad \frac{\partial \mathcal{L}_i}{\partial y_i} = p_i - \lambda_i f_2(x_i, y_i) = 0. \tag{3}$$

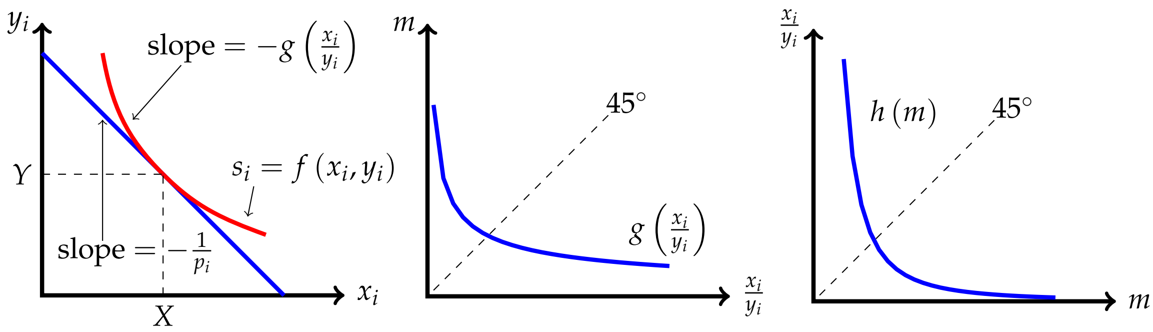
These conditions are also sufficient for a solution to the cost-minimization problem, as the production function is strictly quasiconcave.[10] Hence the conditions in (3), together with the constraint, define the optimal levels of $x_i$ and $y_i$, conditional on $s_i$ and $p_i$. Denote these levels by $X(s_i, p_i)$ and $Y(s_i, p_i)$, respectively.

It will be useful to derive more explicit expressions for $X(s_i, p_i)$ and $Y(s_i, p_i)$. To this end, note that the first-order conditions in (3) can be combined to yield the following condition:

$$\frac{f_1(x_i, y_i)}{f_2(x_i, y_i)} = \frac{1}{p_i}. \tag{4}$$

[9] The analysis here assumes that $s_i > 0$, which implies $p_i > 0$. The case where $s_i = 0$ can be dealt with separately: To achieve a zero score, it is optimal for the contestant to set $x_i = y_i = 0$. If $s_j > 0$ for some $j \neq i$, then setting $x_i = y_i = 0$ yields a zero payoff; if $\mathbf{s_{-i}} = \mathbf{0_{-i}}$, then it yields the payoff $p_i(\mathbf{0})\, v_i$. This will be taken into account in the proof of Proposition 2.

[10] In terms of Figure 1, panel (a), the set of values of $x_i$ and $y_i$ that satisfy $f(x_i, y_i) \geq s_i$ is strictly convex (by the definition of strict quasiconcavity). This guarantees that the point of tangency between the isocost line and the isoquant is unique.




**(a)** Cost minimization. **(b)** Graph of the $g$ function. **(c)** Graph of the $h$ function.

**Figure 1:** Panel (a) illustrates a contestant's cost-minimization problem. The absolute value of the isoquant's slope, known as the marginal rate of technical substitution (MRTS), depends only on the ratio $x_i/y_i$. The function $g$ is defined as the value of the MRTS at $x_i/y_i$. This function is graphed in panel (b), where a particular value of the MRTS is denoted by $m$. The function $h$ is the inverse of $g$ and it is graphed in panel (c) as a function of $m$.

The left-hand side of (4) is the marginal rate of technical substitution (MRTS) between $x_i$ and $y_i$, and the right-hand side is the relative "price" of the two kinds of investment. It is well known that, due to the assumption that $f$ is homogeneous, the MRTS is determined by the ratio $x_i/y_i$, meaning that we can write it as $g(x_i/y_i)$.[11] Moreover, the MRTS is a strictly decreasing function of this ratio, $g'(x_i/y_i) < 0$.[12] We can thus write condition (4) as $g(x_i/y_i) = 1/p_i$ or

$$x_i = y_i h\left(\frac{1}{p_i}\right), \tag{5}$$

where $h$ is the inverse of $g$ (i.e., $h \stackrel{\text{def}}{=} g^{-1}$). In words, the function $h$ tells us which investment ratio $x_i/y_i$ that is consistent with a particular value of the MRTS. Since $g$ is strictly decreasing, so is $h$. The graphs of these two functions are plotted in panels (b) and (c) of Figure 1. The third column of Table 1 indicates which $h$ functions are associated with certain production functions.

We can now use (5) to eliminate $x_i$ from the constraint $s_i = f(x_i, y_i)$.[13] Thereafter, with the help of the resulting expression and (5), we can solve for $y_i$ and $x_i$. We then obtain:

$$Y(s_i, p_i) = \left[\frac{s_i}{f(h(1/p_i), 1)}\right]^{\frac{1}{t}}, \qquad X(s_i, p_i) = Y(s_i, p_i)\, h\left(\frac{1}{p_i}\right). \tag{6}$$

[11]Since the function $f(x_i, y_i)$ is homogeneous of degree $t$, its partial derivatives are homogeneous of degree $t-1$. The MRTS can therefore be written as

$$\frac{f_1(x_i, y_i)}{f_2(x_i, y_i)} = \frac{k^{-(t-1)} f_1(kx_i, ky_i)}{k^{-(t-1)} f_2(kx_i, ky_i)} = \frac{f_1\left(\frac{x_i}{y_i}, 1\right)}{f_2\left(\frac{x_i}{y_i}, 1\right)} \stackrel{\text{def}}{=} g\left(\frac{x_i}{y_i}\right),$$

where the second equality is obtained by setting $k = 1/y_i$.

[12]This follows from the strict quasiconcavity of $f(x_i, y_i)$; cf. panel (a) of Figure 1.

[13]Doing that yields $s_i = f[y_i h(1/p_i), y_i] = y_i^t f[h(1/p_i), 1]$, where the second equality uses the assumption that $f$ is homogeneous of degree $t$.

Since the production function $f(x_i, y_i)$ is assumed to be thrice differentiable, $X(s_i, p_i)$ and $Y(s_i, p_i)$ are also differentiable in $p_i$ at least twice.

### 3.2 Step 2: Choosing the Optimal Score

At step 2 I let contestant $i$ choose the optimal value of the score, acknowledging that here $p_i$ is not a parameter but a function of the score. Contestant $i$'s payoff can be written as

$$\pi_i(\mathbf{s}) = p_i(\mathbf{s}) v_i - C[s_i, p_i(\mathbf{s})], \tag{7}$$

where $p_i(\mathbf{s})$ is given by (1) and where

$$C[s_i, p_i(\mathbf{s})] \overset{\text{def}}{=} p_i(\mathbf{s}) Y[s_i, p_i(\mathbf{s})] + X[s_i, p_i(\mathbf{s})] \tag{8}$$

is contestant $i$'s minimized expected costs, conditional on $s_i$ (and $\mathbf{s_{-i}}$). A Nash equilibrium of the hybrid contest can now be defined as a strategy profile $\mathbf{s}^*$ such that $\pi_i(\mathbf{s}^*) \geq \pi_i\left(s_i, \mathbf{s}^*_{-\mathbf{i}}\right)$ for all $s_i \geq 0$ and all contestants $i$. That is, given that all other contestants choose their scores according to the equilibrium, each contestant $i$ must, at least weakly, prefer her equilibrium score to all other scores.

Before characterizing such an equilibrium, we should address the question of equilibrium existence. It follows from a result due to Reny (1999) that a pure strategy equilibrium of the hybrid contest is guaranteed to exist if (i) each contestant $i$'s strategy set is closed and bounded, (ii) her payoff function is quasiconcave in $s_i$, and (iii) the game is "better-reply secure." Condition (i) can easily be taken care of by, without loss of generality, impose an upper bound on $s_i$ and condition (iii) holds under the assumptions we have already made (see the proof of Proposition 1 for details). However, condition (ii) requires more structure on the model than we have imposed so far. Assumption 1 below will specify a sufficient condition for (ii) to hold.

First, however, define the following elasticities:

$$\eta\left(\frac{1}{p_i}\right) \overset{\text{def}}{=} \frac{f_1\left[h\left(\frac{1}{p_i}\right), 1\right] h\left(\frac{1}{p_i}\right)}{f\left[h\left(\frac{1}{p_i}\right), 1\right]}, \qquad \sigma\left(\frac{1}{p_i}\right) \overset{\text{def}}{=} -\frac{h'\left(\frac{1}{p_i}\right)\frac{1}{p_i}}{h\left(\frac{1}{p_i}\right)}, \qquad \varepsilon_i(\mathbf{s}) \overset{\text{def}}{=} \frac{\partial p_i}{\partial s_i}\frac{s_i}{p_i}.$$

In words, $\eta\left(\frac{1}{p_i}\right)$ is the elasticity of output with respect to $x_i$. We have that $\eta\left(\frac{1}{p_i}\right) \in (0, t)$.[14] The second elasticity, $\sigma\left(\frac{1}{p_i}\right) > 0$, is the elasticity of substitution. This is a measure of how easy or difficult it is for a contestant to substitute one kind of investment for another, while keeping the score variable $s_i$ unchanged. For a Cobb-Douglas production function, $\sigma\left(\frac{1}{p_i}\right) = 1$. For a CES production function, the elasticity can take any positive value but is constant. Finally, $\varepsilon_i(\mathbf{s})$ is the elasticity of the win probability with respect to $s_i$. Our assumptions that $p_i$ is strictly increasing and strictly concave in $s_i$ imply that $\varepsilon_i(\mathbf{s}) \in (0, 1)$.

[14] By Euler's homogeneous function theorem, $xf_1(x,y) + yf_2(x,y) = tf(x,y)$. This implies that $xf_1(x,y)/f(x,y) < t$.

**Assumption 1.** *The production function and the CSF satisfy at least one of the following three sets of conditions:*

*(i)* $t \le 1$ *and* $\varepsilon_i(\mathbf{s})\,\eta\left(\frac{1}{p_i}\right)\sigma\left(\frac{1}{p_i}\right) \le 2$ *(for all i, $p_i$, and* $\mathbf{s}$*);*

*(ii)* $tr \le 1$, $r\eta\left(\frac{1}{p_i}\right)\sigma\left(\frac{1}{p_i}\right) \le 2$, *and*

$$p_i(\mathbf{s}) = \frac{w_i s_i^r}{\sum_{j=1}^{n} w_j s_j^r} \quad \text{(for all } i, p_i, \text{ and } \mathbf{s} \neq \mathbf{0}\text{)}, \tag{9}$$

*where* $r > 0$ *and* $w_i > 0$ *are parameters;*

*(iii)* $p_i(\mathbf{s})$ *is given by (9),* $f(x_i, y_i) = x_i^\alpha y_i^\beta$ *(with* $\alpha > 0$ *and* $\beta > 0$*), and* $\alpha r \le 1$ *(for all i).*

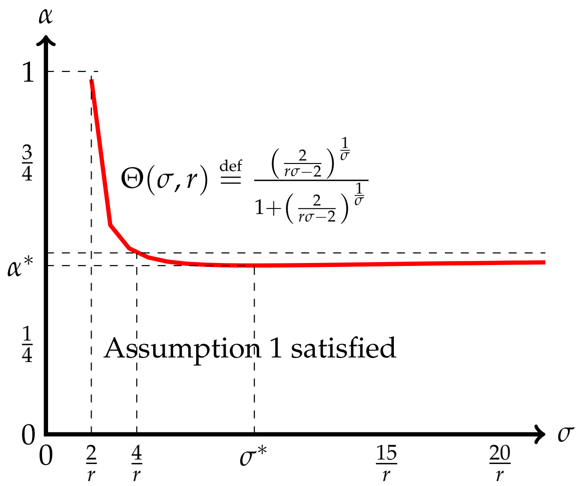
The condition $t \le 1$ in (i) says that the score production function exhibits constant or decreasing returns to scale. If indeed $t \le 1$, then the second condition in (i) is always satisfied for a Cobb-Douglas production function (since then $\sigma = 1$). With a CES production function (still assuming $t \le 1$), the assumption is guaranteed to hold for all $\sigma \in (0, 2]$. The set of conditions (ii) relaxes the requirement that $f(x_i, y_i)$ exhibits non-increasing returns to scale; instead it requires that $p_i(\mathbf{s})$ is of a generalized Tullock form with scale parameter $r$ and that $tr \le 1$. The set of conditions (iii) requires both a generalized Tullock form for the CSF and a Cobb-Douglas production function, but instead replaces condition $tr \le 1$ with $\alpha r \le 1$ (meaning that $\beta$ can be arbitrarily large). This alternative condition holds, for example, in the Cobb-Douglas-Tullock setting with $r = \alpha = \beta = 1$ that is assumed by Haan and Schoonbeek (2003).[15]

**Proposition 1.** ***(Equilibrium existence)*** *Suppose Assumption 1 is satisfied. Then there exists a pure strategy Nash equilibrium of the hybrid contest.*

Proposition 1 represents a significant step forward relative to the analysis in Melkonyan (2013). The condition that is required by the proposition (i.e., Assumption 1) can be satisfied also for arbitrarily large values of the elasticity of substitution as long as, in the production function, the winner-pay investments matter sufficiently much relative to the all-pay investments. This is illustrated in Figure 2, which assumes a CES production technology, constant returns to scale ($t = 1$), and a CSF given by (9). Given CES, the relative importance of all-pay investments in the production function can be measured by a parameter $\alpha$ (see the functional form in Table 1). Figure 2 shows that, if $\alpha$ is small enough ($\alpha \lesssim 0.465$) and if $r \le 1$, then Assumption 1 holds for any $\sigma > 0$. The figure also shows that, in general, it would be misleading to say that $\sigma$ must be sufficiently small for Assumption 1 to be satisfied—for certain $\alpha$'s the assumption is violated for intermediate values of $\sigma$ but satisfied for sufficiently small and large values of this elasticity.[16]

[15]The proofs of Proposition 1 and other results that are not shown in the main text can be found in the appendix. The calculations used for some of the figures are reported in the online appendix (Lagerlöf, 2020).

[16]The reason why Assumption 1 can hold also for large values of $\sigma$ is that the elasticity $\eta\left(\frac{1}{p_i}\right)$ is a function of $\sigma$ and it will, under certain conditions, be small when $\sigma$ is large. In particular, for $\alpha < 1/2$ and $\sigma > 1$, the upper bound of $\eta\left(\frac{1}{p_i}\right)$ equals $\eta(1) = \left(\frac{\alpha}{1-\alpha}\right)^\sigma / \left[\left(\frac{\alpha}{1-\alpha}\right)^\sigma + 1\right]$, which is decreasing in $\sigma$. For further details, see the online appendix (Lagerlöf, 2020).




| $r$ | $\sigma^*$ | $\alpha^*$ | $r$ | $\sigma^*$ | $\alpha^*$ |
|---|---|---|---|---|---|
| .1 | 91.8 | .497 | .6 | 15.3 | .479 |
| .2 | 45.9 | .493 | .7 | 13.1 | .476 |
| .3 | 30.6 | .490 | .8 | 11.5 | .472 |
| .4 | 23.0 | .486 | .9 | 10.2 | .469 |
| .5 | 18.4 | .483 | 1 | 9.2 | .465 |

**Figure 2:** Illustration of Proposition 1. Given a CES production function and a CSF that satisfies (9), with $t = 1$ and $r \leq 1$, Assumption 1 is always satisfied if $\sigma \leq 2/r$; and otherwise it is satisfied if $\alpha \leq \Theta(\sigma, r)$. The function $\Theta(\sigma, r)$ is minimized with respect to $\sigma$ at $\sigma = \sigma^*$, where it takes the value $\alpha^*$.

Now turn to the characterization of equilibrium. The first-order condition associated with the problem of maximizing (7) with respect to $s_i$ can be written as

$$\frac{\partial \pi_i(\mathbf{s})}{\partial s_i} = \frac{\partial p_i(\mathbf{s})}{\partial s_i} v_i - C_1(s_i, p_i) - C_2(s_i, p_i) \frac{\partial p_i(\mathbf{s})}{\partial s_i} \leq 0,$$

with an equality if $s_i > 0$. This inequality can be reformulated by using Shephard's lemma, $C_2(s_i, p_i) = Y[s_i, p_i(\mathbf{s})]$.[17] We thus obtain the following first-order condition for contestant $i$:

$$[v_i - Y(s_i, p_i(\mathbf{s}))] \frac{\partial p_i(\mathbf{s})}{\partial s_i} \leq C_1(s_i, p_i), \tag{10}$$

with an equality if $s_i > 0$. Condition (10) states that, at the optimum, the marginal benefit of a larger $s_i$ must not exceed the marginal cost of a larger $s_i$, where the marginal benefit equals the net value of winning $(v_i - Y[s_i, p_i(\mathbf{s})])$ multiplied by the increase in probability of winning.

**Proposition 2.** ***(Characterization of equilibrium)*** *Suppose Assumption 1 is satisfied. Then* $\mathbf{s}^* = (s_1^*, \ldots, s_n^*)$ *is a pure strategy Nash equilibrium of the hybrid contest if and only if condition (10) holds, with equality if* $s_i^* > 0$*, for each contestant i. Moreover,* $\mathbf{s} = \mathbf{0}$ *is not a Nash equilibrium.*

Once the equilibrium scores have been pinned down by the first-order conditions (10), we can use (1) to determine each contestant's probability of winning and (6) to obtain the investment levels.

[17]This result holds because the effect of a change in $p_i$ on $C[s_i, p_i(\mathbf{s})]$ that goes through $X(s_i, p_i)$ and $Y(s_i, p_i)$ must equal zero, as $x_i$ and $y_i$ have been chosen optimally at step 1 (this is simply an application of the envelope theorem). For a discussion of Shephard's lemma see, for example, Chambers (1988, p. 56 onwards).

## 4 A Symmetric Hybrid Contest

In this section, I derive results for a symmetric hybrid contest: All contestants are ex ante identical (so $v_i = v$) and the CSF is symmetric. In addition, I assume that the CSF is homogeneous.

**Assumption 2.** ***(Symmetry)*** *For all $j \neq i$ and all $a, b \in \Re_+$, $p_i(\mathbf{s})\mid_{(s_i,s_j)=(a,b)} = p_j(\mathbf{s})\mid_{(s_i,s_j)=(b,a)}$.*

**Assumption 3.** ***(Homogeneity of degree $\tilde{t}$)*** *For all $i$ and for all $k > 0$, $p_i(k\mathbf{s}) = k^{\tilde{t}} p_i(\mathbf{s})$.*

By combining Assumption 3 and our previous assumption that $\sum_{i=1}^{n} p_i(\mathbf{s}) = 1$, one can easily show that the CSF function is indeed homogeneous of degree zero ($\tilde{t} = 0$), which means that it is scale invariant. This, in turn, implies that the partial derivative of $p_i(\mathbf{s})$ with respect to $s_i$ is homogeneous of degree $-1$. Now note that, by using the latter result and by evaluating at symmetry, we can write the derivative of the CSF with respect to the own score as

$$\frac{\partial p_i(s,s,\ldots,s)}{\partial s_i} = \frac{\hat{\varepsilon}(n)}{ns}, \quad \text{where} \quad \hat{\varepsilon}(n) \stackrel{\text{def}}{=} \varepsilon_i\,(1,1,\ldots,1)\,.$$

Thus, by imposing symmetry on the first-order condition (10), which here must hold with equality, and by using the expressions in (6) and (8), we have

$$(v - y^*)\,\frac{\hat{\varepsilon}(n)}{ns^*} = C_1\left[s^*, \frac{1}{n}\right] \Leftrightarrow (v - y^*)\, t\hat{\varepsilon}(n) = y^* + nx^*, \tag{11}$$

where $x^* \stackrel{\text{def}}{=} X\left(s^*, \frac{1}{n}\right)$ and $y^* \stackrel{\text{def}}{=} Y\left(s^*, \frac{1}{n}\right)$.[18] The second equality in (11) is linear in $x^*$ and $y^*$, and it is now straightforward to solve for these variables and for $s^*$.

**Proposition 3.** ***(Equilibrium, symmetric model)*** *Suppose Assumptions 1-3 are satisfied and that $v_i = v$ for all $i$. Then, within the family of symmetric equilibria, there is a unique pure strategy Nash equilibrium of the hybrid contest. In this equilibrium, $s^* = f[h(n), 1]\,(y^*)^t$, $x^* = h(n)y^*$, and*

$$y^* = \frac{t\hat{\varepsilon}(n)v}{1 + nh(n) + t\hat{\varepsilon}(n)}. \tag{12}$$

The results in Proposition 3 are a substantial generalization of those in Haan and Schoonbeek (2003) and Melkonyan (2013).[19] The present results hold for any $f$ and $p_i$ functions that are consistent with Assumptions 1-3 and with the model assumptions made in Section 2 (most importantly, that the production function is strictly quasiconcave and homogeneous and that the CSF is strictly concave in the own score). From the results in Proposition 3 we can also, as limit cases, obtain expressions for the equilibrium expenditures in a pure winner-pay contest and a pure all-pay contest. The former is given by $\lim_{h\to 0} y^* = t\hat{\varepsilon}(n)v / \left[1 + t\hat{\varepsilon}(n)\right]$ and the latter

[18] The last step in (11) uses $C_1\left[s^*, \frac{1}{n}\right] = \frac{1}{ts^*} C\left[s^*, \frac{1}{n}\right] = \frac{1}{ts^*}\left(\frac{y^*}{n} + x^*\right)$.

[19] Proposition 3 does not rule out the possibility that there exist asymmetric equilibria in the symmetric model, in addition to the unique symmetric equilibrium described in the proposition. However, by imposing more structure on the model, it is possible to exclude this possibility. In particular, the uniqueness result in Proposition 7 is valid also here, under the additional assumption made there that there are two contestants. This results states that a sufficient condition for equilibrium uniqueness is that $r\eta\left(\frac{1}{p_i}\right)\sigma\left(\frac{1}{p_i}\right) \leq 1$ (for all $p_i \in [0,1]$). That is, one of the conditions in part (i) of Assumption 1 is strengthened somewhat.

equals $\lim_{h\to\infty} x^* = t\widehat{\varepsilon}(n)v/n$. To the best of my knowledge, these closed-form expressions for the equilibrium investments of the symmetric pure all-pay and winner-pay contests are more general than any ones in the previous literature.

Let us now turn to comparative statics. Some of the results reported below require the following additional assumption:

**Assumption 4.** ***(Independence of irrelevant alternatives)*** *For all $i \neq j$ and for all $\mathbf{s}$ with $s_i > 0$, the CSF has the following property:*

$$p_i\left(s_1, \ldots, s_{j-1}, 0, s_{j+1}, \ldots, s_n\right) = \frac{p_i(\mathbf{s})}{1 - p_j(\mathbf{s})}. \tag{13}$$

Assumption 4 says that contestant $i$'s probability of winning when contestant $j$ does not participate is the same as when $j$ indeed participates but is known not to have won.[20] The assumption is always satisfied for $n = 2$. Skaperdas (1996) and Clark and Riis (1998) have shown that Assumptions 2-4 (together with the assumptions about $p_i$ made in the model description) imply that the CSF is of the Tullock form: $p_i(\mathbf{s}) = s_i^r / \sum_{j=1}^n s_j^r$, for $r > 0$. This in turn means that, under those assumptions, we have

$$\widehat{\varepsilon}(n) = \frac{r(n-1)}{n}. \tag{14}$$

In accordance with the notation used in Table 1, let $\alpha$ be a parameter in the production function that increases the relative importance of all-pay investments. In particular, a larger $\alpha$ is associated with a flatter MRTS and thus a larger value of $h$:

$$\frac{\partial h(n)}{\partial \alpha} > 0. \tag{15}$$

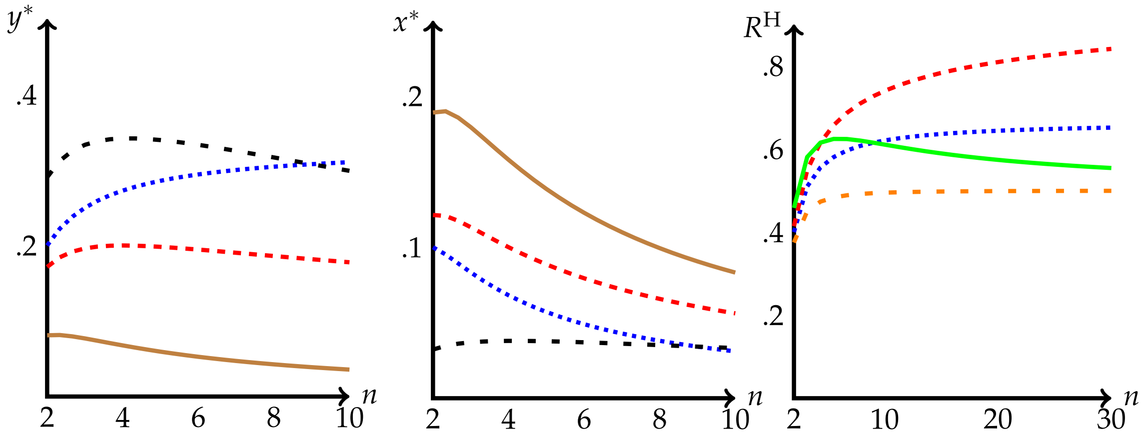
**Proposition 4.** ***(Comparative statics, investment levels)*** *Suppose Assumptions 1-3 are satisfied. Then both $x^*$ and $y^*$ are strictly increasing in $v$ and $t$. Moreover, $x^*$ is strictly increasing and $y^*$ is strictly decreasing in $\alpha$. Suppose in addition that Assumption 4 is satisfied. Then the effects of a larger number of contestants on $x^*$ and $y^*$ are as follows:*

$$\frac{\partial x^*}{\partial n} < 0 \Leftrightarrow \sigma(n) > -\frac{n\,(n-2)\,h\,(n) - 1}{(n-1)\left[1 + \frac{rt(n-1)}{n}\right]}, \qquad \frac{\partial y^*}{\partial n} > 0 \Leftrightarrow \sigma(n) > \frac{n(n-2)h(n) - 1}{(n-1)nh(n)}; \tag{16}$$

*and if $\sigma(n) \geq 1$, then necessarily $\frac{\partial x^*}{\partial n} < 0$ and $\frac{\partial y^*}{\partial n} > 0$.*

The comparative statics results with respect to $v$ and $\alpha$ are straightforward. Similarly, the result about $t$ can easily be understood in light of the fact that this is a returns-to-scale parameter. To understand the comparative statics results with respect to $n$, note that a larger number of contestants in a symmetric equilibrium means a lower probability of winning for any one of them. This lowers the relative cost of investing in $y_i$. As a consequence, whenever it is sufficiently easy to substitute between $x_i$ and $y_i$, we have $\frac{\partial x^*}{\partial n} < 0$ and $\frac{\partial y^*}{\partial n} > 0$. However, if the

[20] Interpreting $s_j = 0$ as "not participating" is natural, given the assumption in the model description that $s_j = 0$ and $s_i > 0$ imply $p_j(\mathbf{s}) = 0$.




**(a)** Graphs of $y^*$. **(b)** Graphs of $x^*$. **(c)** Graphs of $R^H$.

**Figure 3:** Illustration of Propositions 4 and 5. Assuming a CES production function and a Tullock CSF, with $t = r = v = 1$, panels (a) and (b) plot $y^*$ and $x^*$ against $n$ for the following parameter configurations: $(\sigma, \alpha) = (1, \frac{1}{2})$—the blue, dotted curve; $(\sigma, \alpha) = (\frac{1}{2}, \frac{1}{2})$—the red, dashed curve; $(\sigma, \alpha) = (0, \frac{7}{10})$—the brown, solid curve; and $(\sigma, \alpha) = (0, \frac{1}{10})$—the black, loosely dashed curve. Under the same assumptions, panel (c) plots $R^H$ against $n$ for the following parameter configurations: $(\sigma, \alpha) = (1, \frac{1}{2})$—the blue, dotted curve; $(\sigma, \alpha) = (\frac{1}{2}, \frac{1}{2})$—the red, dashed curve; $(\sigma, \alpha) = (2, \frac{9}{10})$—the green, solid curve; and $(\sigma, \alpha) = (2, \frac{1}{2})$—the orange, loosely dashed curve.

elasticity of substitution $\sigma(n)$ is relatively small, then we can have other results. This is easy to see from the relationship

$$\frac{\partial y^*}{\partial n}\frac{n}{y^*} = \sigma(n) + \frac{\partial x^*}{\partial n}\frac{n}{x^*}, \tag{17}$$

which follows immediately from $x^* = h(n)y^*$. In the limit where $\sigma(n) \to 0$, it is clear from (17) that $\frac{\partial x^*}{\partial n}$ and $\frac{\partial y^*}{\partial n}$ must have the same sign. The reason is obvious. As $\sigma(n) \to 0$, the score production function requires $x_i$ and $y_i$ to be used in fixed proportions (a Leontief production technology). It turns out that, by choosing the parameters appropriately, we can make either both derivatives positive (if $f$ is winner-pay intensive) or both negative (if $f$ is all-pay intensive)—at least locally.[21] Panels (a) and (b) of Figure 3 illustrate this.

The total amount of equilibrium expenditures in the symmetric hybrid model is defined as $R^H \stackrel{\text{def}}{=} nC\left[s^*, \frac{1}{n}\right]$. It is interesting to compare the magnitude of $R^H$ to the total equilibrium expenditures in the corresponding pure all-pay contest, which will be denoted by $R^A$. The latter can be obtained from the current framework by, for example, assuming a Cobb-Douglas production function, so that $f(x_i, y_i) = x_i^\alpha y_i^{t-\alpha}$, and then consider the limit $\alpha \to t$. Doing that

[21]It may be surprising that both derivatives can be positive. The reason is that, in a pure winner-pay contest, an individual contestant's investment can be increasing in $n$, as these investments are paid only by the winner and thus the aggregate investments of that contest correspond, in a way, to the individual investments of the pure all-pay contest. (With a lottery function, the individual equilibrium investments in the pure winner-pay contest equal $y^* = (n-1)v/(2n-1)$, which indeed are increasing in $n$.) For a low enough value of $\alpha$, the hybrid contest is sufficiently close to the pure winner-pay contest that it exhibits the same feature.

yields

$$R^A = t\widehat{\varepsilon}(n)v. \tag{18}$$

**Proposition 5.** ***(Total expenditures)*** *Suppose Assumptions 1-3 are satisfied. In the symmetric hybrid contest, the total amount of equilibrium expenditures can be written as:*

$$R^H = \left[1 - \frac{y^*}{v}\right] R^A = \left[\frac{1}{v\left[1+nh(n)\right]} + \frac{1}{R^A}\right]^{-1}. \tag{19}$$

*These expenditures are strictly lower than the total equilibrium expenditures in the corresponding pure all-pay contest, $R^H < R^A$. Moreover, $R^H$ is strictly increasing in $v$, $t$, and $\alpha$. Suppose in addition that Assumption 4 is satisfied. Then $R^H$ is weakly increasing in $n$ if and only if: (i)*

$$\sigma(n) \leq 1 + \frac{4n}{rt\,(n-1)^2}; \tag{20}$$

*or (ii) inequality (20) is violated and $h(n) \notin (\Xi_L, \Xi_H)$, where*

$$\Xi_L \overset{def}{=} \frac{K}{2} - \frac{1}{2n}\sqrt{n^2K^2 - 4},\ \Xi_H \overset{def}{=} \frac{K}{2} + \frac{1}{2n}\sqrt{n^2K^2 - 4},\ \textit{with}\ \ K \overset{def}{=} \frac{rt\,(n-1)^2\left[\sigma(n)-1\right] - 2n}{n^2}.$$

A striking result reported in Proposition 5 is that, for all parameter values, the hybrid contest yields lower total expenditures than the pure all-pay contest (i.e., $R^H < R^A$). We can understand this result by noting that the effective prize that a contestant can win in a hybrid contest is not $v$ (as it is in the pure all-pay contest) but $v - y_i$. All else equal, this lowers the contestant's incentive to invest in $x_i$ and $y_i$ and she will thus be content with a lower value of the score $s_i$. In other words, the contestant's best reply, as implicitly defined by the first-order condition in (10), will shift downwards. As this is true for all contestants, the result is an equilibrium with lower investment levels and expenditures.[22]

Proposition 5 also reports several comparative statics results. As was the case for Proposition 4, the results about $v$ and $t$ are straightforward. The reason why $\partial R^H/\partial \alpha > 0$ is that a larger $\alpha$ makes the hybrid model closer to the pure all-pay contest and this contest is conducive to large expenditures. What about the comparative statics with respect to $n$? In the pure all-pay contest and under Assumption 4, the total expenditures are increasing in this parameter (see eqs. (14) and (18)). A sufficient condition for the same result to hold in the hybrid model is that condition (20) is satisfied, which requires a small enough elasticity of substitution. However, if (20) is violated and if $h(n)$ is neither too large nor too small, then $R^H$ can be decreasing in $n$. The reason is that a larger $n$ makes winner-pay investments less costly in relative terms; this leads to substitution from all-pay to winner-pay investments and thus a larger $y^*$, which lowers the effective value of the prize. The lower value of the prize, in turn, leads to lower total expenditures. The result that $R^H$ can be decreasing in $n$, which was also shown by Melkonyan (2013),[23]

[22] Within a simpler framework, Wärneryd (2000, p. 152) shows in greater detail and with the help of a figure how the best reply shifts downwards in a winner-pay environment relative to an all-pay setting. The interested reader is encouraged to consult Wärneryd's useful discussion.

[23] A similar result, called the *exclusion principle*, has been obtained by Baye, Kovenock and De Vries (1993). However, these authors consider another setting (an all-pay auction) and their result is driven by a different logic (which

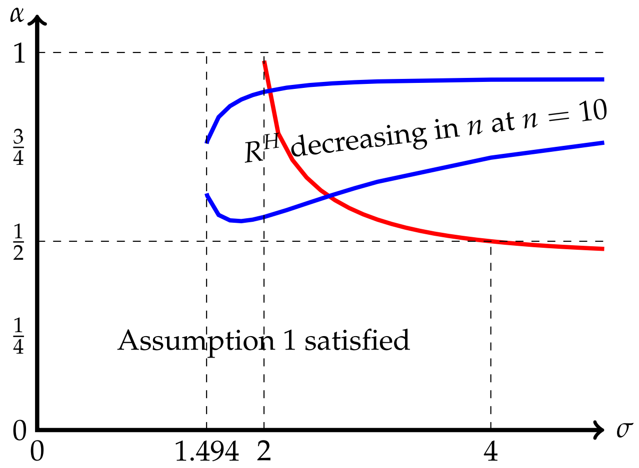



**Figure 4:** In the symmetric model, total expenditures can be decreasing in $n$. The graphs assume a CES production function and $t = r = 1$.

is illustrated in panel (c) of Figure 3. Moreover, for an example with a CES production function, $r = t = 1$, and $n = 10$, Figure 4 indicates where in the $(\alpha, \sigma)$-space that this phenomenon occurs. This figure also confirms that the phenomenon can indeed occur for parameter values for which Assumption 1 is satisfied.

Finally consider the question how the total expenditures, under Assumption 4, evolve as the number of contestants becomes very large. As a benchmark, first note that the limit value of the expenditures in the pure all-pay contest (i.e., $\lim_{n\to\infty} R^A$) equals $trv$; this follows immediately from (14) and (18). Next, from the right-most expression in (19) we see that the way in which the corresponding limit value in the hybrid contest relates to $trv$ depends on the limit value of $nh(n)$. This, in turn, depends on whether $h(n)$ decreases slower or faster than $n$ increases. Proposition 6 summarizes these results.

**Proposition 6.** ***(Limit, total expenditures)*** *Suppose Assumptions 1-4 are satisfied. As $n \to \infty$, the total amount of expenditures in the symmetric hybrid contest can be written as:*

$$\lim_{n\to\infty} R^H = \begin{cases} \frac{trv(1+L)}{trv+1+L} & \text{if } \lim_{n\to\infty} nh(n) \stackrel{def}{=} L \in [0, \infty) \\ trv & \text{if } L = \infty. \end{cases} \tag{21}$$

For a CES production function, we have $nh(n) = \left(\frac{\alpha}{1-\alpha}\right)^{\sigma} n^{1-\sigma}$. This means that, for such a technology, the limit total expenditures equal the ones in the all-pay contest if and only if $\sigma < 1$; for $\sigma \geq 1$, they are strictly lower than the limit total expenditures in the all-pay contest.

involves asymmetry in the contestants' valuations). For further work related to the exclusion principle, see Che and Gale (2000) and Alcalde and Dahm (2010), who study non-deterministic CSFs, and Kirkegaard (2013) who studies a deterministic CSF with incomplete information.

## 5 Asymmetric Hybrid Contests

In this section, I derive results for three asymmetric hybrid contests. In the first model I allow for the possibility that the decision process (i.e., the CSF) is biased in favor of one of the contestants. In the second model I instead let the contestants have different valuations for winning. In the third one I allow for both these kinds of asymmetry, but I let the degree of bias in the CSF be endogenous. Throughout I assume that there are two contestants ($n = 2$) and that Assumptions 1 and 3 hold. In addition, and as already noted, Assumption 4 is automatically satisfied when there are only two contestants. Clark and Riis (1998) have shown that Assumptions 3 and 4 (together with the assumptions about $p_i$ made in the model description) imply the following extended Tullock form of the CSF:

$$p_i(\mathbf{s}) = \frac{w_i s_i^r}{w_1 s_1^r + w_2 s_2^r}, \qquad w_1, w_2 > 0. \tag{22}$$

This functional form implies that the derivative of the win probability with respect to the own score becomes $\partial p_i / \partial s_i = r p_i (1 - p_i)/s_i$. By using this expression and the relationships in (6) and (8), we can write the first-order conditions in (10) as[24]

$$y_i^* = \frac{rtp_i^*(1-p_i^*)v_i}{rtp_i^*(1-p_i^*) + p_i^* + h\left(\frac{1}{p_i^*}\right)}, \qquad \text{for } i = 1, 2. \tag{23}$$

By plugging the equilibrium scores $s_1^*$ and $s_2^*$ into (22), we also obtain the relationship $p_1^* w_2 \left(s_2^*\right)^r = (1 - p_1^*) w_1 \left(s_1^*\right)^r$, which can be restated as $\Upsilon(p_1^*) = 0$, where

$$\Upsilon(p_1) \stackrel{\text{def}}{=} \frac{\frac{w_2 v_2^{rt}}{w_1 v_1^{rt}} p_1 f\left[h\left(\frac{1}{1-p_1}\right), 1\right]^r}{\left[rtp_1(1-p_1) + 1 - p_1 + h\left(\frac{1}{1-p_1}\right)\right]^{rt}} - \frac{(1-p_1) f\left[h\left(\frac{1}{p_1}\right), 1\right]^r}{\left[rtp_1(1-p_1) + p_1 + h\left(\frac{1}{p_1}\right)\right]^{rt}}.$$

We thus obtain the following result.

**Proposition 7.** ***(Characterization and uniqueness of equilibrium)*** *Suppose Assumptions 1 and 3 are satisfied. Moreover, suppose $n = 2$ and that the two contestants are ex ante identical in all respects except that, possibly, $w_1 \neq w_2$ and $v_1 \neq v_2$. Then the equilibrium values of $p_1^*$, $y_1^*$ and $y_2^*$ are determined by the three equations (23) and $\Upsilon(p_1^*) = 0$. The all-pay equilibrium investment levels are obtained from the relationships $x_1^* = y_1^* h(\frac{1}{p_1^*})$ and $x_2^* = y_2^* h(\frac{1}{1-p_1^*})$. The equilibrium is guaranteed to be unique if, for all $p_i \in [0, 1]$, $r\eta\left(\frac{1}{p_i}\right)\sigma\left(\frac{1}{p_i}\right) \leq 1$.*

The condition for uniqueness stated in Proposition 7 is not implied by Assumption 1. Hence, in general we cannot rule out multiplicity of equilibria. The comparative statics analysis presented below will consider an equilibrium in which $\Upsilon'(p_1^*) > 0$ (a stability property). Such an equilibrium always exists under Assumption 1. Thus, if the model has a unique equilibrium, then this indeed satisfies $\Upsilon'(p_1^*) > 0$.

[24]Given that there are only two contestants, both of them must be active in an equilibrium.

### 5.1 A Biased Decision Process

Suppose $v_1 = v_2$ but that we may have $w_1 \neq w_2$. That is, the contestants have the same valuations but the decision process may be biased in favor of contestant 1 (if $w_1 > w_2$) or contestant 2 (if $w_2 > w_1$).

**Proposition 8.** ***(Biased decision process)*** *Suppose Assumptions 1 and 3 are satisfied. Moreover, suppose $n = 2$ and that the two contestants are ex ante identical in all respects except that, possibly, $w_1 \neq w_2$. Then:*

*(i)* $p_1^* > p_2^* \Leftrightarrow y_1^* < y_2^* \Leftrightarrow C(s_1^*, p_1^*) > C(s_2^*, p_2^*)$.

*(ii) Suppose in addition that $\Upsilon'(p_1^*) > 0$. Evaluated at symmetry, contestant 1's equilibrium win probability is strictly increasing in $w_1$ and the equilibrium winner-pay investments of contestant 1 (2, resp.) are strictly decreasing (strictly increasing, resp.) in $w_1$. Moreover,*

$$\frac{\partial x_1^*}{\partial w_1} |_{w_1 = w_2} > 0 \Leftrightarrow \frac{\partial x_2^*}{\partial w_1} |_{w_1 = w_2} < 0 \Leftrightarrow \sigma(2) > \frac{2}{2 + rt}.$$

Part (i) of Proposition 8 says that the contestant who is more likely to win invests less in $y_i$ than her rival does; the reason is that the higher win probability makes the relative cost of winner-pay investments higher, so the contestant does less of it. Part (i) also reports that the expected expenditures of the contestant with the higher win probability are higher than her rival's.

Part (ii) concerns the effect of a small change in $w_1$ on the win probability and on the winner-pay and all-pay investments. To simplify the algebra, the analysis is restricted to the case where the difference between $w_1$ and $w_2$ is small. The results say that, evaluated at symmetry, contestant 1's win probability is increasing in $w_1$, which is probably not very surprising. Similarly, evaluated at symmetry, the winner-pay investments of contestant 1 (2, resp.) go down (up, resp.) as $w_1$ increases. The reason is that, for contestant 1, winner-pay investments become more expensive due to the higher win probability (and vice versa for contestant 2). Moreover, again evaluated at symmetry, the all-pay investments of contestant 1 (2, resp.) are increasing (decreasing, resp.) in $w_1$ if and only if the elasticity of substitution is larger than a particular threshold, which is smaller than unity. That is, if the elasticity of substitution is equal to at least one, then the all-pay investments move in opposite direction to the winner-pay investments. This is simply because, again, each contestant substitutes from one influence channel to another, when their relative costs change. However, for low enough values of the elasticity of substitution, the two investment levels move in the same direction when $w_1$ goes up. For the favored contestant, both decrease; and for her rival, both increase. This may suggest that, for such low values of the elasticity of substitution, the favored contestant's expenditures are lower than those for the rival. Yet the result in part (i) says that this is *not* the case: The favored contestant's expenditures are always higher than her rival's. Apparently, although both $x_i^*$ and $y_i^*$ are lower for the favored contestant, her probability of winning is sufficiently much higher to ensure that the result holds.

### 5.2 Different Valuations

Now suppose that $w_1 = w_2$ but that we may have $v_1 \neq v_2$. That is, the decision process is unbiased but the contestants may have different valuations. We have the following result.

**Proposition 9.** ***(Different valuations)*** *Suppose Assumptions 1 and 3 are satisfied. Moreover, suppose $n = 2$ and that the two contestants are ex ante identical in all respects except that, possibly, $v_1 \neq v_2$. Then:*

*(i)* $p_1^* > p_2^* \Leftrightarrow \frac{y_1^*}{v_1} < \frac{y_2^*}{v_2}$;

*(ii)* $v_1 - y_1^* > v_2 - y_2^* \Leftrightarrow C(s_1^*, p_1^*) > C(s_2^*, p_2^*)$.

Part (i) of Proposition 9 states that a larger win probability is associated with a lower ratio between winner-pay investment and valuation ($y_i^*/v_i$). This differs somewhat from the result in part (i) of Proposition 8. When $v_i$ may vary, as here, it is not necessarily true that the contestant with the higher win probability chooses less winner-pay investments, since this contestant may also have a higher valuation. Part (ii) provides a condition for contestant 1 to contribute more to the expected total expenditures than contestant 2, namely, that the ex post net value of winning ($v_i - y_i^*$) is larger for contestant 1.

### 5.3 Different Valuations and Endogenous Decision Process

Suppose, finally, that contestant 1 may have a higher valuation than contestant 2 ($v_1 \geq v_2$) and that the relationship between $w_1$ and $w_2$ is endogenous. In particular, for any given values of $v_1$, $v_2$, and $w_2$, a principal can freely choose $w_1$ and thus determine the magnitude of the bias in the CSF. The principal's objective is to maximize the expected total amount of equilibrium expenditures. The timing is as follows. First the principal chooses $w_1 \geq 0$; then this choice is observed by the two contestants and, exactly as in the previous subsections, they simultaneously make their all-pay and winner-pay investments. Let $\widehat{w}_1$ denote the value of $w_1$ at a (subgame perfect Nash) equilibrium of the above game. Also, let $\widehat{p}_1$ denote the equilibrium value of $p_1$. What can we say about $\widehat{w}_1$ and $\widehat{p}_1$? I will explore this question under the following assumption:

**Assumption 5.** *The production function is of Cobb-Douglas form: $f(x_i, y_i) = x_i^{\alpha} y_i^{\beta}$, for $\alpha > 0$ and $\beta > 0$.*

Under Assumption 5, and for a given $p_1$, the expected total amount of expenditures can be written as

$$R^H = rtp_1(1-p_1)\frac{r\beta[p_1 v_1 + (1-p_1)v_2] + v_1 + v_2}{[r\beta(1-p_1)+1](r\beta p_1 + 1)} \tag{24}$$

(for a derivation, see the proof of Proposition 10). Moreover, an equilibrium value of $p_1$ satisfies the following equality, which is a special case of $\Upsilon(p_1) = 0$:

$$w_1 = w_2 \left(\frac{p_1}{1-p_1}\right)^{1+r\beta} \left(\frac{r\beta(1-p_1)+1}{r\beta p_1 + 1}\frac{v_2}{v_1}\right)^{rt}. \tag{25}$$

Note that (24) does not depend on $w_1$ directly, only through $p_1$. Thus, $\widehat{w}_1$ can be determined recursively. We first find $\widehat{p}_1$ (by maximizing (24) with respect to $p_1$) and then plug $p_1 = \widehat{p}_1$ into (25) to obtain $\widehat{w}_1$.

**Proposition 10.** ***(Optimal bias)*** *Suppose that Assumptions 1, 3, and 5 are satisfied, that $n = 2$, and that the two contestants are ex ante identical in all respects except that, possibly, $w_1 \neq w_2$ and $v_1 \geq v_2$. Also suppose that $w_1$ is chosen at an ex ante stage so as to maximize the expected total expenditures. Then the equilibrium values of $p_1$ and $w_1$ satisfy: $\widehat{p}_1 = \frac{1}{2}$ and $\widehat{w}_1 = w_2$ if $v_1 = v_2$; and $\widehat{p}_1 > \frac{1}{2}$ if $v_1 > v_2$. Moreover, $\frac{\partial \widehat{p}_1}{\partial v_1} > 0$ and $\frac{\partial \widehat{p}_1}{\partial v_2} < 0$ for all $v_1 \geq v_2$, and $\frac{\partial \widehat{p}_1}{\partial (r\beta)} > 0$ for all $v_1 > v_2$. Finally,*

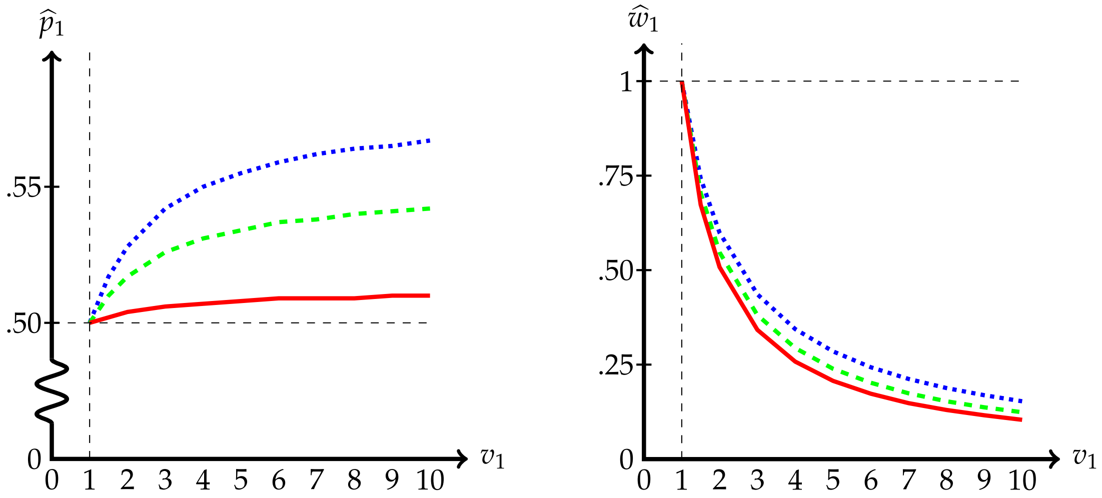
$$\lim_{v_1 \to \infty} \widehat{p}_1 < 1, \qquad \lim_{v_1 \to \infty} \widehat{w}_1 = 0, \qquad \lim_{v_1 \to v_2} \frac{\partial \widehat{w}_1}{\partial v_1} < 0.$$

Proposition 10 says that if $v_1 = v_2$, so that there is no exogenous asymmetry, then the expected total expenditures are maximized by making the CSF unbiased, which also means $\widehat{p}_1 = \frac{1}{2}$. The proposition also says that if $v_1 > v_2$, then the expected total expenditures are maximized by choosing a $w_1$ that makes contestant 1 more likely to win than contestant 2. However, this does not necessarily mean that the bias is in favor of contestant 1, as this contestant also has a higher valuation. On the contrary, for a small difference between $v_1$ and $v_2$, the endogenously chosen bias is necessarily in favor of contestant 2. Likewise, if the difference between the valuations is very large (so $v_1 \to \infty$, while $v_2$ is fixed), then again the bias is in favor of contestant 2. Deriving analytical results for the case where the difference $(v_1 - v_2)$ is neither infinitesimally small nor infinitely large is challenging; however, Figure 5 shows some numerical examples where the optimally chosen bias is in favor of contestant 2 for all $v_1 > v_2$.

Intuitively, the result that $\widehat{p}_1 > \frac{1}{2}$ whenever $v_1 > v_2$ is straightforward to understand. The high-valuation contestant is a more valuable contributor to the expected total expenditures. Therefore, since all-pay investments are more conducive to a high expenditure level, the relative price of all-pay investments should be made lower for contestant 1 than for contestant 2. Hence $\widehat{p}_1 > \frac{1}{2}$. In order to create the outcome $\widehat{p}_1 > \frac{1}{2}$, the principal is helped by the fact that, exogenously, $v_1 > v_2$. This turns out to be more than enough to create the desired difference in win probability—there is no need to, in addition, bias the CSF in favor of contestant 1. Indeed, the effect arising from $v_1 > v_2$ must be alleviated by setting $\widehat{w}_1 < w_2$, i.e., to create a bias *against* the high-valuation contestant. Intuitively, the result that $\widehat{w}_1 < w_2$ does not seem obvious, which raises some questions about its robustness. To explore this further would be interesting but is beyond the scope of the present paper.

## 6 Concluding Remarks

In this paper, I have used a producer theory approach to study contests where the contestants can make both all-pay and winner-pay investments—so-called hybrid contests. This approach allowed for a general analysis that still is very tractable, in particular for the symmetric case. Pure all-pay and winner-pay contests are obtained as limit cases of this setting, where the former limit case is similar to a standard Tullock (1980) contest but more general. Under symmetry,




**(a)** The high-valuation contestant's probability of winning.

**(b)** The weight in the CSF that is assigned to the high-valuation contestant's score.

**Figure 5:** The model with different valuations and endogenous decision process. Both panels assume $t = r = v_2 = 1$ (also recall that $\beta = t - \alpha$). They plot $\widehat{p}_1$ and $\widehat{w}_1$, respectively, against $v_1$ for three different values of $\alpha$: 0.9 (the blue, dotted curve), 0.5 (the green, dashed curve), and 0.1 (the red, solid curve).

the analysis yields a simple closed-form solution, in spite of the general setting.[25]

Thanks to the producer theory approach, I could derive a sufficient condition for equilibrium existence—stated in terms of basic elasticities of the model—that implies that there can be an equilibrium also for arbitrarily large values of the elasticity of substitution. The hybrid contest always gives rise to a smaller amount of expected total expenditures than the corresponding pure all-pay contest. This fact and the contestants' opportunity to substitute were important for some of the comparative statics results. In particular, the results about the relationship between total expenditures and $n$ in Proposition 5 and the optimal-bias result in Proposition 10 are driven by a contestant's incentive to substitute from winner-pay to all-pay investments as the economic environment changes, in conjunction with the fact that all-pay investments are more conducive to large expenditures.

It would be interesting to apply the producer theory approach to other models of contests where the contestants have access to multiple influence channels or where they can choose multi-dimensional efforts (see the literature review in the Introduction). However, in other applications a rival's individual effort levels may possibly matter directly for a contestant's payoff—not only through an aggregator variable like the score in the hybrid model studied here. If so, the approach might not be as helpful as it has been in the present paper.

The analysis in the present paper has given rise to a large number of predictions, which would be desirable to test with the help of experimental or field data. The setting used here should be particularly useful as a basis for such empirical studies, as it is quite general and the

[25]Indeed, also the closed-form solutions for the all-pay and winner-pay limit cases are, to the best of my knowledge, more general than any ones in the previous literature.

analysis has spelled out comparative statics results under a very broad set of circumstances. Yet there are several directions in which the current setting, in future theoretical work, could be extended. Examples of extensions that seem promising and interesting include multi-period settings and/or sequential moves, asymmetric hybrid contests with more than two contestants, and to study contest design questions in broader settings than considered here.

# Appendix

## Proof of Proposition 1

To prove the proposition, we can invoke Theorem 3.1 in Reny (1999), which guarantees the existence of a pure strategy equilibrium under the conditions that the strategy sets are compact, contestant $i$'s payoff function is quasiconcave in $s_i$, and the game is better-reply secure. The first condition is readily taken care of by, without loss of generality, imposing a constraint $s_i \leq \bar{s}$, where $\bar{s}$ is some finite and sufficiently large constant; this ensures that each player's strategy set $[0, \bar{s}] \stackrel{\text{def}}{=} S$ is closed and bounded and thus compact. The requirement that the payoff functions are quasiconcave will be investigated at the end of this proof. To show that the game is better-reply secure, we can rely on Proposition 1 in Bagh and Jofre (2006). This says that a game is better-reply secure if it is *payoff secure* and *weakly reciprocal upper semicontinuous (wrusc)*.[26] We know that, in the hybrid contest, each player's payoff function is continuous everywhere, except possibly at the origin. This means that the potentially problematic issue with showing the two properties is what happens at the point $\mathbf{s} = \mathbf{0}$.

In order to prove that the game is payoff secure at $\mathbf{s} = \mathbf{0}$, we must show that each player can, for every $\epsilon > 0$, secure a payoff of $p_i(\mathbf{0}) v_i - \epsilon$ at $\mathbf{s} = \mathbf{0}$. A player is said to be able to secure a payoff of $p_i(\mathbf{0}) v_i - \epsilon$ at $\mathbf{s} = \mathbf{0}$ if there exists $\tilde{s}_i$ such that $\pi_i(\tilde{s}_i, \mathbf{s}'_{-\mathbf{i}}) \geq p_i(\mathbf{0}) v_i - \epsilon$ for all $\mathbf{s}'_{-\mathbf{i}}$ in some open neighborhood of $\mathbf{0}_{-\mathbf{i}}$. The hybrid contest is indeed payoff secure at $\mathbf{s} = \mathbf{0}$. To see this, note that there exists $\tilde{s}_i > 0$ such that

$$\pi_i(\tilde{s}_i, \mathbf{0}_{-\mathbf{i}}) = p_i(\tilde{s}_i, \mathbf{0}_{-\mathbf{i}}) v_i - C[\tilde{s}_i, p_i(\tilde{s}_i, \mathbf{0}_{-\mathbf{i}})] = v_i - C[\tilde{s}_i, 1] > p_i(\mathbf{0}) v_i. \tag{A1}$$

The second equality in (A1) follows from the assumption that, for any $\tilde{s}_i > 0$, $p_i(\tilde{s}_i, \mathbf{0}_{-\mathbf{i}}) = 1$; the inequality in (A1) follows from (i) the assumption that $p_i(\mathbf{0}) < 1$ and (ii) the fact that $C[\tilde{s}_i, 1]$ can be made arbitrarily small by choosing a $\tilde{s}_i$ close enough to zero. Moreover, $\pi_i$ is continuous at $(\tilde{s}_i, \mathbf{0}_{-\mathbf{i}})$. Therefore, (A1) implies that for every $\epsilon > 0$ and for all $\mathbf{s}'_{-\mathbf{i}}$ in some open neighborhood of $\mathbf{0}_{-\mathbf{i}}$, we have $\pi_i(\tilde{s}_i, \mathbf{s}'_{-\mathbf{i}}) \geq p_i(\mathbf{0}) v_i - \epsilon$.

The *graph* of the game is defined as $\Gamma = \{(\mathbf{s}, \pi_1, \cdots, \pi_n) \in S^n \times \Re^n \mid \pi_i(\mathbf{s}) = \pi_i, \forall i\}$. The closure of $\Gamma$ is denoted by $\overline{\Gamma}$. The *frontier* of $\Gamma$, denoted by Fr $\Gamma$, is defined as the set of points that are in $\overline{\Gamma}$ but not in $\Gamma$. In order to prove that the game is wrusc, we must show that for any $(\mathbf{s}, \beta_1, \dots, \beta_n)$ in the frontier of the game, there is a player $i$ and $\tilde{s}_i$ such that $\pi_i(\tilde{s}_i, \mathbf{s}'_{-\mathbf{i}}) > \beta_i$. The game is indeed wrusc. To verify this, first note that, since the origin is the only point of discontinuity, any point in Fr $\Gamma$ must be of the form $(\mathbf{0}, \gamma_1 v_1, \cdots, \gamma_n v_n)$, where for some $\mathbf{s}^\tau \to \mathbf{0}$ and every $i$, we have $\lim_{\tau \to \infty} \pi_i(\mathbf{s}^\tau) = \gamma_i v_i$. We must also have $\sum_{i=1}^n \gamma_i = 1$. Hence, for some $i$, $\gamma_i < 1$. Suppose, without loss of generality, that $\gamma_1 < 1$. Because $\lim_{s_1 \to 0} \pi_i(s_1, \mathbf{0}_{-\mathbf{1}}) = v_1$, there exists $\tilde{s}_i > 0$ such that $\pi_i(\tilde{s}_i, \mathbf{0}_{-\mathbf{1}}) > \gamma_1 v_1$.

To prove the proposition, it remains to show that, under the conditions stated there, player $i$'s payoff function is quasiconcave in $s_i$. I will do this by showing that $\frac{\partial^2 \pi_i}{\partial s_i^2} < 0$ at any point where $\frac{\partial \pi_i}{\partial s_i} = 0$. From the analysis in the main text, it follows that we can write the derivative of contestant $i$'s payoff function with respect to $s_i$ as $\frac{\partial \pi_i}{\partial s_i} = [v_i - Y(s_i, p_i(\mathbf{s}))] \frac{\partial p_i}{\partial s_i} - C_1(s_i, p_i)$. Differentiating again yields

$$\frac{\partial^2 \pi_i}{\partial s_i^2} = -\left[Y_1(s_i, p_i) + Y_2(s_i, p_i) \frac{\partial p_i}{\partial s_i}\right] \frac{\partial p_i}{\partial s_i} + [v_i - Y(s_i, p_i)] \frac{\partial^2 p_i}{\partial s_i^2} - C_{11}(s_i, p_i) - C_{12}(s_i, p_i) \frac{\partial p_i}{\partial s_i}.$$

Now note that $C_{12}(s_i, p_i) = C_{21}(s_i, p_i) = Y_1(s_i, p_i)$. For a value of $s_i$ for which $\frac{\partial \pi_i}{\partial s_i} = 0$ holds, we also have $v_i - Y(s_i, p_i) = \frac{C_1(s_i, p_i)}{\partial p_i / \partial s_i}$. Moreover, $C_1(s_i, p_i) = \left[p_i + h\left(\frac{1}{p_i}\right)\right] Y_1(s_i, p_i)$ and

$$C_{11}(s_i, p_i) = \left[p_i + h\left(\frac{1}{p_i}\right)\right] Y_{11}(s_i, p_i) = \frac{1-t}{t s_i}\left[p_i + h\left(\frac{1}{p_i}\right)\right] Y_1(s_i, p_i)$$

[26] The proof below that the hybrid contest has those two properties will follow the proof in Example 3 of Bagh and Jofre (2006) fairly closely.

(cf. (6) and (8)). Therefore, evaluated at a value of $s_i$ where $\frac{\partial \pi_i}{\partial s_i}=0$, the second-derivative can be written

$$\frac{\partial^2 \pi_i}{\partial s_i^2}\mid_{\frac{\partial \pi_i}{\partial s_i}=0}= -\left[2Y_1(s_i,p_i)+Y_2(s_i,p_i)\frac{\partial p_i}{\partial s_i}\right]\frac{\partial p_i}{\partial s_i}+\left[\frac{\partial^2 p_i/\partial s_i^2}{\partial p_i/\partial s_i}-\frac{1-t}{ts_i}\right]\left[p_i+h\left(\frac{1}{p_i}\right)\right]Y_1(s_i,p_i). \tag{A2}$$

The expression in (A2) is strictly negative if and only if

$$\left[2\frac{Y_1(s_i,p_i)s_i}{Y(s_i,p_i)}+\frac{Y_2(s_i,p_i)p_i}{Y(s_i,p_i)}\frac{\partial p_i}{\partial s_i}\frac{s_i}{p_i}\right]\frac{\partial p_i}{\partial s_i}>\left[\frac{\partial^2 p_i/\partial s_i^2}{\partial p_i/\partial s_i}-\frac{1-t}{ts_i}\right]\left[p_i+h\left(\frac{1}{p_i}\right)\right]\frac{Y_1(s_i,p_i)s_i}{Y(s_i,p_i)}. \tag{A3}$$

Now note that $\frac{Y_1(s_i,p_i)s_i}{Y(s_i,p_i)}=\frac{1}{t}$ and

$$\begin{aligned}\frac{Y_2(s_i,p_i)p_i}{Y(s_i,p_i)} &= -\frac{1}{t}(s_i)^{\frac{1}{t}}\left[f\left(h\left(\frac{1}{p_i}\right),1\right)\right]^{-\frac{1}{t}-1}f_1\left[h\left(\frac{1}{p_i}\right),1\right]h'\left(\frac{1}{p_i}\right)\left(\frac{-1}{p_i^2}\right)\times p_i\left[\frac{s_i}{f(h(1/p_i),1)}\right]^{-\frac{1}{t}}\\ &= \frac{1}{t}\frac{f_1\left[h\left(\frac{1}{p_i}\right),1\right]h\left(\frac{1}{p_i}\right)}{f\left(h\left(\frac{1}{p_i}\right),1\right)}\times\frac{h'\left(\frac{1}{p_i}\right)\frac{1}{p_i}}{h\left(\frac{1}{p_i}\right)}=-\frac{\eta\left(\frac{1}{p_i}\right)\sigma\left(\frac{1}{p_i}\right)}{t}.\end{aligned}$$

Inequality (A3) can therefore be written as

$$\left[\frac{2}{t}-\frac{\eta\left(\frac{1}{p_i}\right)\sigma\left(\frac{1}{p_i}\right)\varepsilon_i(\mathbf{s})}{t}\right]\frac{\partial p_i}{\partial s_i}>\left[\frac{\partial^2 p_i/\partial s_i^2}{\partial p_i/\partial s_i}-\frac{1-t}{ts_i}\right][p_i+h(1/p_i)]\frac{1}{t}$$

or, equivalently, as

$$\eta\left(\frac{1}{p_i}\right)\sigma\left(\frac{1}{p_i}\right)\varepsilon_i(\mathbf{s})<2-\left[\frac{\partial^2 p_i/\partial s_i^2}{\partial p_i/\partial s_i}-\frac{1-t}{ts_i}\right]\frac{\left[p_i+h\left(\frac{1}{p_i}\right)\right]}{\partial p_i/\partial s_i}. \tag{A4}$$

The last term in the above inequality is strictly negative for all $t\leq 1$. Therefore, a sufficient condition for (A4) to hold is that $\eta\left(\frac{1}{p_i}\right)\sigma\left(\frac{1}{p_i}\right)\varepsilon_i(\mathbf{s})\leq 2$. This proves the claim for part (i) of Assumption 1. In order to prove the claim for part (ii), note that the derivative of the CSF in (9) can be written as $\frac{\partial p_i}{\partial s_i}=rp_i(1-p_i)/s_i$, and the second-derivative is given by $\frac{\partial^2 p_i}{\partial s_i^2}=rp_i(1-p_i)[r(1-2p_i)-1]/s_i^2$. Thus, the term in square brackets in (A4) becomes

$$\frac{\partial^2 p_i/\partial s_i^2}{\partial p_i/\partial s_i}-\frac{1-t}{ts_i}=\frac{r(1-2p_i)-1}{s_i}-\frac{1-t}{ts_i}=\frac{tr(1-2p_i)-1}{ts_i},$$

which is non-positive for all $p_i$ if $tr\leq 1$. Moreover, $\varepsilon_i(\mathbf{s})=r(1-p_i)\leq r$. Hence the result follows. Finally consider part (iii). The additional Cobb-Douglas assumption means that we can write the last term in (A4) as

$$\left[\frac{\partial^2 p_i/\partial s_i^2}{\partial p_i/\partial s_i}-\frac{1-t}{ts_i}\right]\left[\frac{p_i+h\left(\frac{1}{p_i}\right)}{\partial p_i/\partial s_i}\right]=\left[\frac{tr(1-2p_i)-1}{ts_i}\right]\left[\frac{p_i+\frac{\alpha}{\beta}p_i}{rp_i(1-p_i)/s_i}\right]=\frac{tr(1-2p_i)-1}{r\beta(1-p_i)}.$$

Moreover, the left-hand side of (A4) simplifies to $\eta\left(\frac{1}{p_i}\right)\sigma\left(\frac{1}{p_i}\right)\varepsilon_i(\mathbf{s})=\alpha r(1-p_i)$. Inequality (A4) therefore becomes

$$\alpha r(1-p_i)<2-\frac{tr(1-2p_i)-1}{r\beta(1-p_i)}\Leftrightarrow\alpha\beta r^2(1-p_i)^2<2r\beta(1-p_i)-tr(1-2p_i)+1.$$

This inequality is most stringent at $p_i=0$ (and it is strictly less stringent for higher values of $p_i$). It therefore suffices if the inequality holds weakly when evaluated at $p_i=0$:

$$\alpha\beta r^2\leq 2r\beta-tr+1=r\beta-\alpha r+1\Leftrightarrow 0\leq r\beta(1-\alpha r)+1-\alpha r\Leftrightarrow\alpha r\leq 1,$$

which gives us the result. □

## Proof of Proposition 2

First consider the claim in the last sentence of the proposition. To verify that $\mathbf{s}=\mathbf{0}$ cannot be a Nash equilibrium, note that $\pi_i(\mathbf{0})=p_i(\mathbf{0})v_i<v_i$. Moreover, by assumption we have $p_i(s_i,\mathbf{0_{-i}})=1$ for any $s_i>0$. Therefore, if contestant $i$ were to deviate from $s_i=0$ to some $s_i>0$, her payoff would equal $\pi_i(s_i,\mathbf{0_{-i}})=v_i-C[s_i,1]$. But $C[s_i,1]$ can be made arbitrarily small by choosing an $s_i$ close enough to zero and, hence, for such an $s_i$ the deviation is profitable.

We can thus conclude that in any equilibrium, $\mathbf{s} \neq \mathbf{0}$. Moreover, we know that each contestant's payoff function is continuous and differentiable for all $\mathbf{s} \neq \mathbf{0}$. In addition, Assumption 1 takes care of the second-order condition. It follows that the analysis in the text that precedes the first-order condition (10) is valid and that this first-order condition indeed characterizes the equilibria of the model. $\square$

## Proof of Proposition 3

Under symmetry, the expression in (5) can be written as $x^* = h(n) y^*$. Plugging this into (11) and then solving for $y^*$ yields (12). The solution to this linear equation system is unique, and so the model has a unique equilibrium within the family of symmetric equilibria. The expression for $s^*$ is obtained by plugging $h(1/p_i) = h(n)$ and $y_i = y^*$ into the equality $s_i = y_i^t f[h(1/p_i), 1]$, which was derived in footnote 13. $\square$

## Proof of Proposition 4

The claims about $v$, $t$, and $\alpha$ are straightforward to verify, so the calculations are omitted. Consider the condition for $y^*$ to be strictly increasing in $n$. Differentiating the expression for $y^*$ in (12), we have

$$\frac{\partial y^*}{\partial n} = \frac{\widehat{\varepsilon}'(n)[1 + nh(n) + t\widehat{\varepsilon}(n)] - \widehat{\varepsilon}(n)[h(n) + nh'(n) + t\widehat{\varepsilon}'(n)]}{(tv)^{-1}[1 + nh(n) + t\widehat{\varepsilon}(n)]^2} > 0 \Leftrightarrow \widehat{\varepsilon}'(n)[1 + nh(n)] > \widehat{\varepsilon}(n)\left[h(n) + nh'(n)\right].$$

Differentiating (14), we obtain $\widehat{\varepsilon}'(n) = r/n^2$. Using this and (14) in the second inequality above yields $1 + nh(n) > n(n-1)[h(n) + nh'(n)] = n(n-1)h(n)[1 - \sigma(n)]$, which simplifies to the condition in (16). Next consider to the condition for $x^*$ to be strictly decreasing in $n$. We have $x^* = h(n) y^*$, where $y^*$ is given by (12). Differentiating yields

$$\frac{\partial x^*}{\partial n} = \frac{[\widehat{\varepsilon}'(n) h(n) + \widehat{\varepsilon}(n) h'(n)][1 + nh(n) + t\widehat{\varepsilon}(n)] - \widehat{\varepsilon}(n) h(n)[h(n) + nh'(n) + t\widehat{\varepsilon}'(n)]}{(tv)^{-1}[1 + nh(n) + t\widehat{\varepsilon}(n)]^2} < 0 \Leftrightarrow$$

$$\left[\widehat{\varepsilon}'(n) h(n) + \widehat{\varepsilon}(n) h'(n)\right][1 + nh(n)] + t[\widehat{\varepsilon}(n)]^2 h'(n) < \widehat{\varepsilon}(n) h(n)\left[h(n) + nh'(n)\right].$$

Dividing through by $\widehat{\varepsilon}(n)$ and using $\widehat{\varepsilon}'(n)/\widehat{\varepsilon}(n) = 1/[n(n-1)]$, the inequality simplifies to

$$\left[\frac{h(n)}{n(n-1)} + h'(n)\right][1 + nh(n)] + t\widehat{\varepsilon}(n) h'(n) < h(n)\left[h(n) + nh'(n)\right]$$

or, equivalently, $h(n)[1 - (n-1)\sigma(n)][1 + nh(n)] - t\widehat{\varepsilon}(n)(n-1)h(n)\sigma(n) < n(n-1)[h(n)]^2[1 - \sigma(n)]$, which simplifies to the condition in (16). Finally consider the claim that $\sigma(n) \geq 1$ is sufficient for both conditions in (16) to hold. Substituting $\frac{n-2}{n-1}$ (which is smaller than unity) for $\sigma(n)$ in the condition for $\frac{\partial y^*}{\partial n}$ in (16) yields

$$\frac{n-2}{n-1} > \frac{n(n-2)h(n) - 1}{n(n-1)h(n)} \Leftrightarrow (n-2)nh(n) > n(n-2)h(n) - 1 \Leftrightarrow 1 > 0,$$

which always holds. And substituting 1 for $\sigma(n)$ in the condition for $\frac{\partial x^*}{\partial n}$ in (16) yields

$$1 > -\frac{n(n-2)h(n) - 1}{(n-1)[1 + t\widehat{\varepsilon}(n)]} \Leftrightarrow (n-1)[1 + t\widehat{\varepsilon}(n)] > -n(n-2)h(n) + 1 \Leftrightarrow n - 2 + t\widehat{\varepsilon}(n)(n-1) > -n(n-2)h(n),$$

which again always holds. $\square$

## Proof of Proposition 5

The first equality in (19) follows immediately from (11) and (18), since $nC\left[s^*, \frac{1}{n}\right] = y^* + nx^*$. To verify the second equality, note that

$$\left(1 - \frac{y^*}{v}\right) R^A = \left(1 - \frac{R^A/v}{1 + nh(n) + R^A/v}\right) R^A = \frac{R^A[1 + nh(n)]v}{[1 + nh(n)]v + R^A} = \left[\frac{1}{[1 + nh(n)]v} + \frac{1}{R^A}\right]^{-1},$$

where the first equality uses (12) and (18). The claim that $R^H < R^A$ follows immediately from (19) and $y^* > 0$. The claims about $v$, $t$, and $\alpha$ are straightforward to verify, so the calculations are omitted. Consider the condition for $R^{\mathrm{H}}$

to be weakly increasing in $n$. By differentiating the right-most expression for $R^{\mathrm{H}}$ in (19), we have

$$\frac{\partial R^H}{\partial n}=-\left[\frac{1}{v\left[1+nh\left(n\right)\right]}+\frac{1}{R^A}\right]^{-2}\left[-\frac{h\left(n\right)+nh'\left(n\right)}{v\left[1+nh\left(n\right)\right]^2}-\frac{\partial R^A/\partial n}{\left(R^A\right)^2}\right]\geq 0\Leftrightarrow\frac{\partial R^A/\partial n}{\left(R^A\right)^2}\geq-\frac{h\left(n\right)\left[1-\sigma\left(n\right)\right]}{v\left[1+nh\left(n\right)\right]^2}.$$

By differentiating the expression in (18) (also using (14)), we obtain $\partial R^A/\partial n=tvr/n^2$. By plugging this and the expression for $R^{\mathrm{A}}$ in (18) (combined with (14)) into the above inequality and then rewriting, we have

$$rt\left(n-1\right)^2\left[\sigma\left(n\right)-1\right]h\left(n\right)\leq\left[1+nh\left(n\right)\right]^2=1+2nh\left(n\right)+n^2h\left(n\right)^2\Leftrightarrow h\left(n\right)^2-Kh\left(n\right)\geq-\frac{1}{n^2},\tag{A5}$$

where $K$ is defined in Proposition 5. Since $h\left(n\right)>0$, this inequality always holds if $K\leq 0$. Suppose $K>0$. Then the left-hand side is negative for all $h\left(n\right)<K$, and it is minimized at $h\left(n\right)=K/2$. Evaluating inequality (A5) at $h\left(n\right)=K/2$ yields

$$-\frac{K^2}{4}\geq-\frac{1}{n^2}\Leftrightarrow K\leq\frac{2}{n}\Leftrightarrow\sigma\left(n\right)\leq 1+\frac{4n}{tr\left(n-1\right)^2}.\tag{A6}$$

Thus if (A6) holds, then (A5) is always satisfied. If (A6) is violated, then also (A5) is violated for values of $h\left(n\right)$ between the two roots of (A5). Solving for these roots (by completing the square), we have:

$$h\left(n\right)^2-Kh\left(n\right)=-\frac{1}{n^2}\Leftrightarrow\left[h\left(n\right)-\frac{K}{2}\right]^2=\frac{n^2K^2}{4n^2}-\frac{4}{4n^2}\Leftrightarrow h\left(n\right)=\frac{K}{2}\pm\frac{1}{2n}\sqrt{n^2K^2-4}.$$

Thus, total expenditures are increasing in $n$ if and only if (i) inequality (A6) holds or (ii) inequality (A6) is violated and $h\left(n\right)\notin\left(\Xi_L,\Xi_H\right)$, where $\Xi_L$ and $\Xi_H$ are defined in Proposition 5. □

## Proof of Proposition 7

The first-order condition in (10) can be written as

$$\left(v_i-y_i^*\right)\frac{rp_i^*\left(1-p_i^*\right)}{s_i^*}=\frac{1}{ts_i^*}C\left(s_i^*,p_i^*\right)\Leftrightarrow rt\left(v_i-y_i^*\right)p_i^*\left(1-p_i^*\right)=\left[p_i^*+h\left(\frac{1}{p_i^*}\right)\right]y_i^*,\tag{A7}$$

where the relationships $C_1\left(s_i^*,p_i^*\right)=\frac{1}{ts_i^*}C\left(s_i^*,p_i^*\right)$ and $C\left(s_i^*,p_i^*\right)=\left[p_i^*+h\left(\frac{1}{p_i^*}\right)\right]y_i^*$ were used. By solving (A7) for $y_i^*$, we obtain (23). The remaining parts of the characterization claim are either shown in the main text or straightforward. It remains to prove the uniqueness claim. Note that the equilibrium is defined recursively: The only endogenous variable in the equality $\Upsilon\left(p_1\right)=0$ is $p_1$; moreover, given a value of $p_1^*$, the winner-pay investments $y_1^*$ and $y_2^*$ are uniquely determined by (23). To prove the claim, it thus suffices to show that if $r\eta\left(\frac{1}{p_i}\right)\sigma\left(\frac{1}{p_i}\right)\leq 1$ for all $p_i\in[0,1]$, then the equation $\Upsilon\left(p_1\right)=0$ has a unique root. A sufficient condition for this, in turn, is that $\Upsilon\left(p_1\right)$ is strictly increasing (by Proposition 1, we know that the equation has at least one root). The equation $\Upsilon\left(p_1\right)=0$ can equivalently be written as $\widehat{\Upsilon}\left(p_1\right)=0$, where

$$\begin{aligned}\widehat{\Upsilon}\left(p_1\right) &= \ln\left[\frac{w_2v_2^{rt}}{w_1v_1^{rt}}\right]+\ln p_1+r\ln f\left[h\left(\frac{1}{1-p_1}\right),1\right]+rt\ln\left[rtp_1\left(1-p_1\right)+p_1+h\left(\frac{1}{p_1}\right)\right]\\ &\quad-\ln\left(1-p_1\right)-r\ln f\left[h\left(\frac{1}{p_1}\right),1\right]-rt\ln\left[rtp_1\left(1-p_1\right)+1-p_1+h\left(\frac{1}{1-p_1}\right)\right].\end{aligned}$$

Differentiating with respect to $p_1$ yields

$$\begin{aligned}
\widehat{\Upsilon}'(p_1) &= \frac{1}{p_1} + \frac{rf_1\left[h\left(\frac{1}{1-p_1}\right),1\right]h'\left(\frac{1}{1-p_1}\right)\frac{1}{(1-p_1)^2}}{f\left[h\left(\frac{1}{1-p_1}\right),1\right]} + \frac{rt\left[rt\left(1-2p_1\right)+1-h'\left(\frac{1}{p_1}\right)\frac{1}{p_1^2}\right]}{rtp_1\left(1-p_1\right)+p_1+h\left(\frac{1}{p_1}\right)} \\
&\quad + \frac{1}{1-p_1} + \frac{rf_1\left[h\left(\frac{1}{p_1}\right),1\right]h'\left(\frac{1}{p_1}\right)\frac{1}{p_1^2}}{f\left[h\left(\frac{1}{p_1}\right),1\right]} - \frac{rt\left[rt\left(1-2p_1\right)-1+h'\left(\frac{1}{1-p_1}\right)\frac{1}{(1-p_1)^2}\right]}{rtp_1\left(1-p_1\right)+1-p_1+h\left(\frac{1}{1-p_1}\right)} \\
&= \frac{1}{p_1\left(1-p_1\right)} - \frac{r\eta\left(\frac{1}{1-p_1}\right)\sigma\left(\frac{1}{1-p_1}\right)}{1-p_1} - \frac{r\eta\left(\frac{1}{p_1}\right)\sigma\left(\frac{1}{p_1}\right)}{p_1} \\
&\quad + \frac{rt\left[rt\left(1-2p_1\right)+1-h'\left(\frac{1}{p_1}\right)\frac{1}{p_1^2}\right]}{rtp_1\left(1-p_1\right)+p_1+h\left(\frac{1}{p_1}\right)} - \frac{rt\left[rt\left(1-2p_1\right)-1+h'\left(\frac{1}{1-p_1}\right)\frac{1}{(1-p_1)^2}\right]}{rtp_1\left(1-p_1\right)+1-p_1+h\left(\frac{1}{1-p_1}\right)}. \qquad \text{(A8)}
\end{aligned}$$

Under the assumption that $r\eta\left(\frac{1}{p_i}\right)\sigma\left(\frac{1}{p_i}\right) \leq 1$ for all $p_i$, the first line of (A8) is non-negative. The second line of (A8) is strictly positive if

$$\begin{aligned}
\frac{rt\left[rt\left(1-2p_1\right)\right]}{rtp_1\left(1-p_1\right)+p_1+h\left(\frac{1}{p_1}\right)} - \frac{rt\left[rt\left(1-2p_1\right)\right]}{rtp_1\left(1-p_1\right)+1-p_1+h\left(\frac{1}{1-p_1}\right)} &\geq 0 \Leftrightarrow \\
\left(1-2p_1\right)\left[1-p_1+h\left(\frac{1}{1-p_1}\right)-p_1-h\left(\frac{1}{p_1}\right)\right] = \left(1-2p_1\right)^2 + \left(1-2p_1\right)\int_{\frac{1}{p_1}}^{\frac{1}{1-p_1}} h'(z)\,dz &\geq 0.
\end{aligned}$$

But, since $h' < 0$, the last inequality holds for all $p_1 \in [0,1]$ (with equality if, and only if, $p_1 = 0.5$). $\square$

## Proof of Proposition 8

Under the assumption that $v_1 = v_2$, (A7) simplifies to $rt\left(v-y_i^*\right)p_i^*\left(1-p_i^*\right) = \left[p_i^*+h\left(\frac{1}{p_i^*}\right)\right]y_i^*$. Since the expression in square brackets is strictly increasing in $p_i^*$ and since $p_1^*\left(1-p_1^*\right) = p_2^*\left(1-p_2^*\right)$, the equality implies that $p_1^* > p_2^* \Leftrightarrow y_1^* < y_2^*$. Moreover, since $\left[p_i^*+h\left(\frac{1}{p_i^*}\right)\right]y_i^* = C\left(s_i^*, p_i^*\right)$, it also implies that $y_1^* < y_2^* \Leftrightarrow C\left(s_1^*, p_1^*\right) > C\left(s_2^*, p_2^*\right)$. This proves part (i). Next turn to part (ii). By taking logs of the three equations (23) and $\Upsilon(p_1^*) = 0$, we have

$$\ln r + \ln t + \ln\left(v_1 - y_1^*\right) + \ln p_1^* + \ln\left(1-p_1^*\right) = \ln\left[p_1^* + h\left(\frac{1}{p_1^*}\right)\right] + \ln y_1^*, \qquad \text{(A9)}$$

$$\ln r + \ln t + \ln\left(v_2 - y_2^*\right) + \ln p_1^* + \ln\left(1-p_1^*\right) = \ln\left[1-p_1^* + h\left(\frac{1}{1-p_1^*}\right)\right] + \ln y_2^*, \qquad \text{(A10)}$$

$$\ln p_1^* + \ln w_2 + r\ln f\left[h\left(\frac{1}{1-p_1^*}\right),1\right] + rt\ln y_2^* = \ln\left(1-p_1^*\right) + \ln w_1 + r\ln f\left[h\left(\frac{1}{p_1^*}\right),1\right] + rt\ln y_1^*. \qquad \text{(A11)}$$

Now set $v_1 = v_2 = v$ in (A9) and (A10). Then differentiate (A9) with respect to $w_1$:

$$\begin{aligned}
-\frac{1}{v-y_1^*}\frac{\partial y_1^*}{\partial w_1} + \left[\frac{1}{p_1^*} - \frac{1}{1-p_1^*}\right]\frac{\partial p_1^*}{\partial w_1} &= \frac{1-h'\left(\frac{1}{p_1^*}\right)\frac{1}{(p_1^*)^2}}{p_1^*+h\left(\frac{1}{p_1^*}\right)}\frac{\partial p_1^*}{\partial w_1} + \frac{1}{y_1^*}\frac{\partial y_1^*}{\partial w_1} \Leftrightarrow \\
\left[\frac{1-2p_1^*}{p_1^*\left(1-p_1^*\right)}\right]\frac{\partial p_1^*}{\partial w_1} &= \frac{\frac{1}{p_1^*}\left[p_1^*+\sigma\left(\frac{1}{p_1^*}\right)h\left(\frac{1}{p_1^*}\right)\right]}{p_1^*+h\left(\frac{1}{p_1^*}\right)}\frac{\partial p_1^*}{\partial w_1} + \frac{v}{y_1^*\left(v-y_1^*\right)}\frac{\partial y_1^*}{\partial w_1} \Leftrightarrow \\
\left[\frac{1-2p_1^*}{1-p_1^*}\right]\frac{\partial p_1^*}{\partial w_1}\frac{w_1}{p_1^*} &= \frac{p_1^*+\sigma\left(\frac{1}{p_1^*}\right)h\left(\frac{1}{p_1^*}\right)}{p_1^*+h\left(\frac{1}{p_1^*}\right)}\frac{\partial p_1^*}{\partial w_1}\frac{w_1}{p_1^*} + \frac{v}{v-y_1^*}\frac{\partial y_1^*}{\partial w_1}\frac{w_1}{y_1^*} \Leftrightarrow \\
\left[\frac{1-2p_1^*-A_1\left(1-p_1^*\right)}{1-p_1^*}\right]\frac{\partial p_1^*}{\partial w_1}\frac{w_1}{p_1^*} &= \frac{v}{v-y_1^*}\frac{\partial y_1^*}{\partial w_1}\frac{w_1}{y_1^*}, \qquad \text{(A12)}
\end{aligned}$$

where $A_1 \stackrel{\text{def}}{=} \left[p_1^* + \sigma\left(\frac{1}{p_1^*}\right) h\left(\frac{1}{p_1^*}\right)\right] / \left[p_1^* + h\left(\frac{1}{p_1^*}\right)\right]$. Similarly, by differentiating (A10) with respect to $w_1$ and then rewriting, we obtain the following equality (the derivation is very similar to the one above):

$$\left[\frac{1 - 2p_1^* + A_2 p_1^*}{1 - p_1^*}\right] \frac{\partial p_1^*}{\partial w_1} \frac{w_1}{p_1^*} = \frac{v}{v - y_2^*} \frac{\partial y_2^*}{\partial w_1} \frac{w_1}{y_2^*}, \tag{A13}$$

where $A_2 \stackrel{\text{def}}{=} \left[1 - p_1^* + \sigma\left(\frac{1}{1-p_1^*}\right) h\left(\frac{1}{1-p_1^*}\right)\right] / \left[1 - p_1^* + h\left(\frac{1}{1-p_1^*}\right)\right]$. Finally differentiate (A11) with respect to $w_1$:

$$\frac{1}{p_1^*} \frac{\partial p_1^*}{\partial w_1} + \frac{r f_1\left[h\left(\frac{1}{1-p_1^*}\right), 1\right] h'\left(\frac{1}{1-p_1^*}\right) \frac{1}{(1-p_1^*)^2}}{f\left[h\left(\frac{1}{1-p_1^*}\right), 1\right]} \frac{\partial p_1^*}{\partial w_1} + rt \frac{1}{y_2^*} \frac{\partial y_2^*}{\partial w_1}$$

$$= \quad -\frac{1}{1 - p_1^*} \frac{\partial p_1^*}{\partial w_1} + \frac{1}{w_1} - \frac{r f_1\left[h\left(\frac{1}{p_1^*}\right), 1\right] h'\left(\frac{1}{p_1^*}\right) \frac{1}{(p_1^*)^2}}{f\left[h\left(\frac{1}{p_1^*}\right), 1\right]} \frac{\partial p_1^*}{\partial w_1} + rt \frac{1}{y_1^*} \frac{\partial y_1^*}{\partial w_1} \Leftrightarrow$$

$$\frac{1}{p_1^*\left(1 - p_1^*\right)} \frac{\partial p_1^*}{\partial w_1} - \frac{r\eta\left(\frac{1}{1-p_1^*}\right)\sigma\left(\frac{1}{1-p_1^*}\right)}{1 - p_1^*} \frac{\partial p_1^*}{\partial w_1} + rt \frac{1}{y_2^*} \frac{\partial y_2^*}{\partial w_1} = \frac{1}{w_1} + \frac{r\eta\left(\frac{1}{p_1^*}\right)\sigma\left(\frac{1}{p_1^*}\right)}{p_1^*} \frac{\partial p_1^*}{\partial w_1} + rt \frac{1}{y_1^*} \frac{\partial y_1^*}{\partial w_1} \Leftrightarrow$$

$$\left[\frac{1 - r\eta\left(\frac{1}{1-p_1^*}\right)\sigma\left(\frac{1}{1-p_1^*}\right) p_1^* - r\eta\left(\frac{1}{p_1^*}\right)\sigma\left(\frac{1}{p_1^*}\right)\left(1 - p_1^*\right)}{p_1^*\left(1 - p_1^*\right)}\right] \frac{\partial p_1^*}{\partial w_1} + rt \frac{1}{y_2^*} \frac{\partial y_2^*}{\partial w_1} = \frac{1}{w_1} + rt \frac{1}{y_1^*} \frac{\partial y_1^*}{\partial w_1} \Leftrightarrow$$

$$\frac{1 - B}{1 - p_1^*} \frac{\partial p_1^*}{\partial w_1} \frac{w_1}{p_1^*} + rt \frac{\partial y_2^*}{\partial w_1} \frac{w_1}{y_2^*} = 1 + rt \frac{\partial y_1^*}{\partial w_1} \frac{w_1}{y_1^*}, \text{ where } B \stackrel{\text{def}}{=} r\eta\left(\frac{1}{1 - p_1^*}\right)\sigma\left(\frac{1}{1 - p_1^*}\right) p_1^* + r\eta\left(\frac{1}{p_1^*}\right)\sigma\left(\frac{1}{p_1^*}\right)\left(1 - p_1^*\right). \tag{A14}$$

Now evaluate the three equations (A12)-(A14) above at $w_1 = w_2$ (all expressions below, until the end of the proof, are evaluated at symmetry, even though this is not everywhere explicitly indicated):

$$-A \frac{\partial p_1^*}{\partial w_1} \frac{w_1}{p_1^*} = \frac{v}{v - y^*} \frac{\partial y_1^*}{\partial w_1} \frac{w_1}{y^*}, \quad A \frac{\partial p_1^*}{\partial w_1} \frac{w_1}{p_1^*} = \frac{v}{v - y^*} \frac{\partial y_2^*}{\partial w_1} \frac{w_1}{y^*}, \quad 2(1 - B) \frac{\partial p_1^*}{\partial w_1} \frac{w_1}{p_1^*} + rt \frac{\partial y_2^*}{\partial w_1} \frac{w_1}{y^*} = 1 + rt \frac{\partial y_1^*}{\partial w_1} \frac{w_1}{y^*},$$

where $A \stackrel{\text{def}}{=} [1 + 2\sigma(2) h(2)] / [1 + 2h(2)]$, $B \stackrel{\text{def}}{=} r\eta(2)\sigma(2)$, and $y^*$ is the common value of $y_1^*$ and $y_2^*$ when evaluated at symmetry. Solving this equation system yields

$$\frac{\partial p_1^*}{\partial w_1} \frac{w_1}{p_1^*} = \frac{1}{2\left[1 - B + rtA\frac{v - y^*}{v}\right]}, \quad \frac{\partial y_1^*}{\partial w_1} \frac{w_1}{y^*} = -\frac{A\frac{v - y^*}{v}}{2\left[1 - B + rtA\frac{v - y^*}{v}\right]}, \quad \frac{\partial y_2^*}{\partial w_1} \frac{w_1}{y^*} = \frac{A\frac{v - y^*}{v}}{2\left[1 - B + rtA\frac{v - y^*}{v}\right]}. \tag{A15}$$

From these results, most of the comparative statics claims follow. To prove the only remaining claim, the one about the all-pay investments, note that the relationship $x_1^* = h\left(\frac{1}{p_1^*}\right) y_1^*$ implies that (at symmetry)

$$\frac{\partial x_1^*}{\partial w_1} \frac{w_1}{x^*} = \sigma(2) \frac{\partial p_1^*}{\partial w_1} \frac{w_1}{p_1^*} + \frac{\partial y_1^*}{\partial w_1} \frac{w_1}{y^*} = \frac{\sigma(2) - A\frac{v - y^*}{v}}{2\left[1 - B + rtA\frac{v - y^*}{v}\right]}, \tag{A16}$$

where $x^*$ is the common value of $x_1^*$ and $x_2^*$ when evaluated at symmetry. Thus,

$$\frac{\partial x_1^*}{\partial w_1} > 0 \Leftrightarrow \sigma(2) > A\frac{v - y^*}{v} = \frac{1 + 2\sigma(2) h(2)}{1 + 2h(2) + \frac{tr}{2}} \Leftrightarrow \sigma(2) > \frac{1}{1 + \frac{tr}{2}} = \frac{2}{2 + tr},$$

where the first equality is obtained by using (23). Similarly, from the relationship $x_2^* = h\left(\frac{1}{1-p_1^*}\right) y_2^*$ we have (at symmetry)

$$\frac{\partial x_2^*}{\partial w_1} \frac{w_1}{x^*} = -\sigma(2) \frac{\partial p_1^*}{\partial w_1} \frac{w_1}{p_1^*} + \frac{\partial y_2^*}{\partial w_1} \frac{w_1}{y^*} = \frac{-\sigma(2) + A\frac{v - y^*}{v}}{2\left[1 - B + rtA\frac{v - y^*}{v}\right]}, \tag{A17}$$

which has the opposite sign to (A16). □

## Proof of Proposition 9

Equation (A7) can be restated as $rt\left(v_i - y_i^*\right) p_i^*\left(1 - p_i^*\right) = C\left(s_i^*, p_i^*\right)$. Since $p_1^*\left(1 - p_1^*\right) = p_2^*\left(1 - p_2^*\right)$, the equality implies that $v_1 - y_1^* > v_2 - y_2^* \Leftrightarrow C\left(s_1^*, p_1^*\right) > C\left(s_2^*, p_2^*\right)$. We can also write (A7) as $rt\left(\frac{v_i}{y_i^*} - 1\right) p_i^*\left(1 - p_i^*\right) =$

$p_i^* + h\left(\frac{1}{p_i^*}\right)$. Since the right-hand side is strictly increasing in $p_i^*$ and since $p_1^*\left(1-p_1^*\right) = p_2^*\left(1-p_2^*\right)$, the equality implies that $p_1^* > p_2^* \Leftrightarrow \frac{y_1^*}{v_1} < \frac{y_2^*}{v_2}$. □

## Proof of Proposition 10

The Cobb-Douglas specification (Assumption 5) implies $h(m) = \frac{\alpha}{\beta} m^{-1}$. By using this in (23), we get

$$v_1 - y_1^* = \frac{v_1\left(p_1 + \frac{\alpha}{\beta} p_1\right)}{rtp_1(1-p_1) + p_1 + \frac{\alpha}{\beta} p_1} = \frac{v_1 \frac{t}{\beta}}{rt(1-p_1) + \frac{t}{\beta}} = \frac{v_1}{r\beta(1-p_1)+1}, \tag{A18}$$

$$v_2 - y_2^* = \frac{v_2\left[1 - p_1 + \frac{\alpha}{\beta}(1-p_1)\right]}{rtp(1-p_1) + 1 - p_1 + \frac{\alpha}{\beta}(1-p_1)} = \frac{v_2 \frac{t}{\beta}}{rtp_1 + \frac{t}{\beta}} = \frac{v_2}{r\beta p_1 + 1}. \tag{A19}$$

Moreover, it follows from (A7) that the expected total equilibrium expenditures can be written as $R^H = rtp_1(1-p_1) \times \left[\left(v_1 - y_1^*\right) + \left(v_2 - y_2^*\right)\right]$. Plugging (A18) and (A19) into this expression yields the expression for $R^H$ stated in (24). Next, taking logs of both sides of (24), we can write

$$\begin{aligned} \ln R^H \quad = \quad & \ln rt + \ln p_1 + \ln(1-p_1) + \ln\left\{r\beta\left[p_1 v_1 + (1-p_1) v_2\right] + v_1 + v_2\right\} \\ & - \ln\left[r\beta(1-p_1)+1\right] - \ln(r\beta p_1 + 1) \end{aligned}$$

Differentiating yields:

$$\begin{aligned} \frac{\partial \ln R^H}{\partial p_1} &= \frac{1}{p_1} - \frac{1}{1-p_1} + \frac{r\beta(v_1 - v_2)}{r\beta\left[p_1 v_1 + (1-p_1) v_2\right] + v_1 + v_2} + \frac{r\beta}{r\beta(1-p_1)+1} - \frac{r\beta}{r\beta p_1 + 1} \\ &= \frac{1-2p_1}{p_1(1-p_1)} + \frac{r\beta(v_1-v_2)}{r\beta\left[p_1 v_1 + (1-p_1) v_2\right] + v_1 + v_2} + \frac{(r\beta)^2(2p_1 - 1)}{(r\beta)^2 p_1(1-p_1) + r\beta + 1} \\ &= \frac{(1-2p_1)(r\beta+1)}{p_1(1-p_1)\left[(r\beta)^2 p_1(1-p_1) + r\beta + 1\right]} + \frac{r\beta(v_1-v_2)}{r\beta\left[p_1 v_1 + (1-p_1) v_2\right] + v_1 + v_2} \overset{\text{def}}{=} F(p_1). \end{aligned} \tag{A20}$$

First consider the case $v_1 = v_2$. Then it is clear from inspection that (A20) is positive for $p_1 < \frac{1}{2}$ and negative for $p_1 > \frac{1}{2}$. Hence, $\widehat{p}_1 = \frac{1}{2}$. Next consider the case $v_1 > v_2$. The derivative w.r.t. $p_1$ of the first term in (A20) is strictly negative:

$$\frac{\partial T(p_1)}{\partial p_1} = (r\beta+1)\frac{-2p_1(1-p_1)\left[(r\beta)^2 p_1(1-p_1) + r\beta + 1\right] - (1-2p_1)^2\left[2(r\beta)^2 p_1(1-p_1) + r\beta + 1\right]}{p_1^2(1-p_1)^2\left[(r\beta)^2 p_1(1-p_1) + r\beta + 1\right]^2} < 0, \tag{A21}$$

where $T$ is short-hand notation for the first term in (A20). Moreover, by inspection, the second term in (A20) is strictly decreasing in $p_1$. Therefore, $\partial^2 \ln R^H / \partial p_1^2 < 0$. Moreover, evaluated at $p_1 = \frac{1}{2}$, the expression in (A20) is strictly positive, whereas it approaches $-\infty$ as $p_1 \to 1$. It follows that $\widehat{p}_1 \in \left(\frac{1}{2}, 1\right)$. In particular, for any $v_1 \geq v_2$, $\widehat{p}_1$ is characterized by $F(\widehat{p}_1) = 0$.

One can verify that $F(p_1)$ is strictly increasing in $v_1$ and strictly decreasing in $v_2$. Hence, $\partial\widehat{p}_1/\partial v_1 > 0$ and $\partial\widehat{p}_1/\partial v_2 < 0$ (the former result will also follow from computations shown below). In order to do comparative statics w.r.t. $r\beta$, differentiate the first term of $F(p_1)$ w.r.t. $r\beta$:

$$\begin{aligned} & \frac{(1-2p_1)}{p_1(1-p_1)} \frac{(r\beta)^2 p_1(1-p_1) + r\beta + 1 - (r\beta+1)\left[2r\beta p_1(1-p_1) + 1\right]}{\left[(r\beta)^2 p_1(1-p_1) + r\beta + 1\right]^2} \\ = \; & \frac{-(1-2p_1)}{p_1(1-p_1)} \frac{p_1(1-p_1) r\beta\left[2(r\beta+1) - r\beta\right]}{\left[(r\beta)^2 p_1(1-p_1) + r\beta + 1\right]^2} = -\frac{(1-2p_1) r\beta(r\beta+2)}{\left[(r\beta)^2 p_1(1-p_1) + r\beta + 1\right]^2}. \end{aligned}$$

Then differentiate the second term of $F(p_1)$ w.r.t. $r\beta$:

$$(v_1 - v_2)\frac{r\beta\left[p_1 v_1 + (1-p_1) v_2\right] + v_1 + v_2 - r\beta\left[p_1 v_1 + (1-p_1) v_2\right]}{\left\{r\beta\left[p_1 v_1 + (1-p_1) v_2\right] + v_1 + v_2\right\}^2} = \frac{(v_1 - v_2)(v_1 + v_2)}{\left\{r\beta\left[p_1 v_1 + (1-p_1) v_2\right] + v_1 + v_2\right\}^2}.$$

Thus, if $v_1 = v_2$, then $F(p_1)$ is constant w.r.t. $r\beta$ and $\partial\widehat{p}_1/\partial(r\beta) = 0$. And if $v_1 > v_2$, then $F(p_1)$ is strictly increasing in $v_1$ and $\partial\widehat{p}_1/\partial(r\beta) > 0$.

Given the Cobb-Douglas specification in Assumption 5, the equation $\Upsilon(p_1^*) = 0$, which defines the equilibrium value of $p_1$, becomes

$$\frac{\frac{w_2 v_2^{rt}}{w_1 v_1^{rt}} p_1 \left[\left(\frac{\alpha}{\beta}\right)^{\alpha}(1-p_1)^{\alpha}\right]^{r}}{\left[rtp_1(1-p_1)+1-p_1+\frac{\alpha}{\beta}(1-p_1)\right]^{rt}} = \frac{(1-p_1)\left[\left(\frac{\alpha}{\beta}\right)^{\alpha} p_1^{\alpha}\right]^{r}}{\left[rtp_1(1-p_1)+p_1+\frac{\alpha}{\beta}p_1\right]^{rt}} \Leftrightarrow$$

$$\frac{\frac{w_2 v_2^{rt}}{w_1 v_1^{rt}} p_1 (1-p_1)^{r\alpha}}{(1-p_1)^{rt}\left(rtp_1+1+\frac{\alpha}{\beta}\right)^{rt}} = \frac{p_1^{r\alpha}(1-p_1)}{p_1^{rt}\left[rt(1-p_1)+1+\frac{\alpha}{\beta}\right]^{rt}} \Leftrightarrow \frac{\frac{w_2 v_2^{rt}}{w_1 v_1^{rt}} p_1^{1+r\beta}}{\left(rtp_1+\frac{t}{\beta}\right)^{rt}} = \frac{(1-p_1)^{1+r\beta}}{\left[rt(1-p_1)+\frac{t}{\beta}\right]^{rt}} \Leftrightarrow$$

$$w_1 = w_2\left[\frac{p_1}{1-p_1}\right]^{1+r\beta}\left[\frac{r(1-p_1)+\frac{1}{\beta}}{rp_1+\frac{1}{\beta}}\frac{v_2}{v_1}\right]^{rt} = w_2\left[\frac{p_1}{1-p_1}\right]^{1+r\beta}\left[\frac{r\beta(1-p_1)+1}{r\beta p_1+1}\frac{v_2}{v_1}\right]^{rt},$$

which gives us (25). The result that $\lim_{v_1\to\infty}\widehat{p}_1 < 1$ follows from inspection of (A20): $F(\widehat{p}_1) = 0$ is inconsistent with $\lim_{v_1\to\infty}\widehat{p}_1 = 1$. Similarly, the result that $\lim_{v_1\to\infty}\widehat{w}_1 = 0$ follows from (25) and the fact that $\lim_{v_1\to\infty}\widehat{p}_1 < 1$.

It remains to prove the last limit result stated in the proposition. In order to do that, we must first derive the value of $\lim_{v_1\to v_2}\partial\widehat{p}_1/\partial v_1$. To this end, differentiate both sides of $F(\widehat{p}_1) = 0$, to obtain:

$$\frac{\partial T(\widehat{p}_1)}{\partial p_1}\frac{\partial\widehat{p}_1}{\partial v_1} - \frac{(r\beta)^2(v_1-v_2)^2}{\left[r\beta\left[\widehat{p}_1 v_1+(1-\widehat{p}_1)v_2\right]+v_1+v_2\right]^2}\frac{\partial\widehat{p}_1}{\partial v_1} + \frac{r\beta\left\{r\beta\left[\widehat{p}_1 v_1+(1-\widehat{p}_1)v_2\right]+v_1+v_2-(v_1-v_2)(r\beta\widehat{p}_1+1)\right\}}{\left[r\beta\left[\widehat{p}_1 v_1+(1-\widehat{p}_1)v_2\right]+v_1+v_2\right]^2} = 0. \tag{A22}$$

The numerator of the last term simplifies to $r\beta(r\beta+2)v_2 > 0$. Since we also know, from above, that $\partial T(\widehat{p}_1)/\partial p_1 < 0$, it follows that $\partial\widehat{p}_1/\partial v_1 > 0$. Next, take the limit $v_1 \to v_2$ of both sides of (A22):

$$\left[\lim_{v_1\to v_2}\frac{\partial T(\widehat{p}_1)}{\partial p_1}\right]\left[\lim_{v_1\to v_2}\frac{\partial\widehat{p}_1}{\partial v_1}\right] + \frac{r\beta(r\beta+2)v_2}{\left[r\beta v_2+v_2+v_2\right]^2} = 0.$$

From (A21) we also have

$$\lim_{v_1\to v_2}\frac{\partial T(\widehat{p}_1)}{\partial p_1} = (r\beta+1)\frac{-8\left[\left(\frac{r\beta}{2}\right)^2+r\beta+1\right]}{\left[\left(\frac{r\beta}{2}\right)^2+r\beta+1\right]^2} = -\frac{8(r\beta+1)}{\left(\frac{r\beta}{2}\right)^2+r\beta+1} = -\frac{32(r\beta+1)}{(r\beta+2)^2}.$$

Thus, $\lim_{v_1\to v_2}\frac{\partial\widehat{p}_1}{\partial v_1} = \left[-\frac{r\beta(r\beta+2)v_2}{(r\beta+2)^2 v_2^2}\right] / \left[-\frac{32(r\beta+1)}{(r\beta+2)^2}\right] = \frac{r\beta(r\beta+2)}{32(r\beta+1)v_2}$. We can now prove the last limit result stated in the proposition. Take logs of (25) and evaluate at $p = \widehat{p}_1$:

$$\ln\widehat{w}_1 = \ln w_2 - (1+r\beta)\ln(1-\widehat{p}_1) + (1+r\beta)\ln\widehat{p}_1 - rt\ln(r\beta\widehat{p}_1+1) + rt\ln\left[r\beta(1-\widehat{p}_1)+1\right] - rt\ln v_1 + rt\ln v_2.$$

Differentiate both sides w.r.t. $v_1$:

$$\begin{aligned}\frac{1}{\widehat{w}_1}\frac{\partial\widehat{w}_1}{\partial v_1} &= \left[\frac{1+r\beta}{1-\widehat{p}_1} + \frac{1+r\beta}{\widehat{p}_1} - \frac{tr^2\beta}{r\beta\widehat{p}_1+1} - \frac{tr^2\beta}{r\beta(1-\widehat{p}_1)+1}\right]\frac{\partial\widehat{p}_1}{\partial v_1} - \frac{rt}{v_1} \\ &= \left[\frac{1+r\beta}{(1-\widehat{p}_1)\widehat{p}_1} - \frac{tr^2\beta(r\beta+2)}{(r\beta\widehat{p}_1+1)\left[r\beta(1-\widehat{p}_1)+1\right]}\right]\frac{\partial\widehat{p}_1}{\partial v_1} - \frac{rt}{v_1}.\end{aligned}$$

Next take the limit $v_1 \to v_2$ of both sides:

$$\lim_{v_1\to v_2}\left[\frac{1}{\widehat{w}_1}\right]\left[\lim_{v_1\to v_2}\frac{\partial\widehat{w}_1}{\partial v_1}\right] = \left[4(1+r\beta) - \frac{tr^2\beta(r\beta+2)}{\left(\frac{r\beta}{2}+1\right)\left(\frac{r\beta}{2}+1\right)}\right]\left[\lim_{v_1\to v_2}\frac{\partial\widehat{p}_1}{\partial v_1}\right] - \frac{rt}{v_2} \Leftrightarrow$$

$$\begin{aligned}\frac{1}{w_2}\lim_{v_1\to v_2}\frac{\partial\widehat{w}_1}{\partial v_1} &= 4\left[(1+r\beta) - \frac{tr^2\beta(r\beta+2)}{(r\beta+2)^2}\right]\frac{r\beta(r\beta+2)}{32(r\beta+1)v_2} - \frac{rt}{v_2} \\ &= \frac{r\beta(r\beta+2)}{8v_2} - \frac{rt}{v_2} - \frac{tr^3\beta^2}{8(r\beta+1)v_2} = \frac{r\beta(r\beta+2)}{8v_2} - \frac{rt\left[8(r\beta+1)+(r\beta)^2\right]}{8(r\beta+1)v_2}.\end{aligned}$$

Thus,

$$\lim_{v_1 \to v_2} \frac{\partial \widehat{w}_1}{\partial v_1} < 0 \Leftrightarrow \frac{r\beta\left(r\beta+2\right)}{8v_2} < \frac{rt\left[8\left(r\beta+1\right)+\left(r\beta\right)^2\right]}{8\left(r\beta+1\right)v_2} \Leftrightarrow \frac{\beta}{\alpha+\beta} < \frac{8\left(r\beta+1\right)+\left(r\beta\right)^2}{\left(r\beta+2\right)\left(r\beta+1\right)} = \frac{5r\beta+6}{\left(r\beta+2\right)\left(r\beta+1\right)}+1,$$

which always holds. □

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