

Supplementary Material to “Hybrid All-Pay and Winner-Pay Contests”

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1. Introduction

In this Supplementary Material, which is not meant to be published, I provide some proofs that were omitted from Lagerlöf (2019). In particular, I here show the calculations that were used for Figures 2, 4, and 5 of that paper.

2. Proofs of Results Not Proven in the Paper

2.1. Calculations Used for Figure 2

Assume a CES production function, a CSF of the generalized Tullock form (as in eq. (9) in Lagerlöf, 2019), and that $t = 1$ and $r \leq 1$. Under these assumptions, condition (i) in Assumption 1 is satisfied for all $\sigma \leq 1$. Thus suppose that $\sigma > 1$. Table 1 in Lagerlöf (2019) tells us that, under the stated assumptions, $\eta\left(\frac{1}{p_i}\right) = \left(\frac{\alpha}{1-\alpha}\right)^\sigma p_i^{\sigma-1} / \left[\left(\frac{\alpha}{1-\alpha}\right)^\sigma p_i^{\sigma-1} + 1\right]$. For $\sigma > 1$, this expression is strictly increasing in p_i . Therefore, since $p_i \leq 1$, an upper bound on $\eta\left(\frac{1}{p_i}\right)$ is given by $\left(\frac{\alpha}{1-\alpha}\right)^\sigma / \left[\left(\frac{\alpha}{1-\alpha}\right)^\sigma + 1\right]$. It follows that condition (i) in Assumption 1 (i.e., $r\eta\left(\frac{1}{p_i}\right)\sigma \leq 2$) is satisfied for all $p_i \in [0, 1]$ if

$$r \frac{\left(\frac{\alpha}{1-\alpha}\right)^\sigma}{\left(\frac{\alpha}{1-\alpha}\right)^\sigma + 1} \sigma \leq 2 \Leftrightarrow (r\sigma - 2) \left(\frac{\alpha}{1-\alpha}\right)^\sigma \leq 2.$$

This inequality is satisfied for all $\sigma \leq 2/r$. Suppose $\sigma > 2/r$. Then the inequality can be rewritten as

$$\alpha \leq \frac{\left(\frac{2}{r\sigma-2}\right)^{\frac{1}{\sigma}}}{1 + \left(\frac{2}{r\sigma-2}\right)^{\frac{1}{\sigma}}} \stackrel{\text{def}}{=} \Theta(\sigma, r).$$

This is the function that is graphed in Figure 2 in Lagerlöf (2019). Note that the derivative of $\Theta(\sigma, r)$ has the same sign as the derivative of $\frac{1}{\sigma} [\ln 2 - \ln(r\sigma - 2)]$. Differentiating the latter expression with respect to σ yields

$$\frac{\ln(r\sigma - 2) - \ln 2 - \frac{r\sigma}{r\sigma-2}}{\sigma^2}, \quad (\text{S1})$$

which clearly is negative for all $r\sigma \leq 4$. Moreover, the numerator in (S1) is increasing in σ and for sufficiently large values of σ the numerator is positive. Thus, for all $\sigma \leq 4/r$, $\Theta(\sigma, r)$ is downward-sloping and there is a unique σ , such that $\sigma > 4/r$, for which $\Theta(\sigma, r)$ is minimized. This value of σ , which I denote by $\sigma = \sigma^*$, is characterized by $\ln(r\sigma^* - 2) - \ln 2 - \frac{r\sigma^*}{r\sigma^*-2} = 0$. The values of σ^* shown in the table in Figure 2 in Lagerlöf (2019) are obtained by, using Maple, solving this equation for different r values. The table also shows the associated minimized values values of $\Theta(\sigma, r)$, denoted by $\alpha^* = \Theta(\sigma^*, r)$. \square

2.2. Calculations Used for Figure 4

In Figure 4 in Lagerlöf (2019) there are two graphs that indicate the part of the parameter space where R^H is decreasing in n (at $n = 10$). I here describe how these graphs were obtained. By assuming a CES production function (which implies $h(n) = \left(\frac{\alpha}{(1-\alpha)n}\right)^\sigma$) and by setting $t = r = 1$, we can write

$$\begin{aligned} h(n) &> \Xi_L \Leftrightarrow \left(\frac{\alpha}{1-\alpha}\right)^\sigma n^{-\sigma} > \frac{(n-1)^2(\sigma-1) - 2n}{2n^2} - \frac{1}{2n} \sqrt{\frac{[(n-1)^2(\sigma-1) - 2n]^2}{n^2} - 4} \Leftrightarrow \\ \left(\frac{\alpha}{1-\alpha}\right)^\sigma &> \frac{(n-1)^2(\sigma-1) - 2n - \sqrt{[(n-1)^2(\sigma-1) - 2n]^2 - 4n^2}}{2n^{2-\sigma}} \\ &= \frac{[(n-1)^2(\sigma-1) - 2n]^2 - \left[[(n-1)^2(\sigma-1) - 2n]^2 - 4n^2\right]}{2n^{2-\sigma} \left[(n-1)^2(\sigma-1) - 2n + \sqrt{[(n-1)^2(\sigma-1) - 2n]^2 - 4n^2} \right]} \\ &= \frac{2n^\sigma}{(n-1)^2(\sigma-1) - 2n + \sqrt{[(n-1)^2(\sigma-1) - 2n]^2 - 4n^2}} \Leftrightarrow \\ \frac{\alpha}{1-\alpha} &> \frac{2^{\frac{1}{\sigma}} n}{\left[(n-1)^2(\sigma-1) - 2n + \sqrt{[(n-1)^2(\sigma-1) - 2n]^2 - 4n^2} \right]^{\frac{1}{\sigma}}} \Leftrightarrow \\ \alpha &> \frac{2^{\frac{1}{\sigma}} n}{2^{\frac{1}{\sigma}} n + \left[(n-1)^2(\sigma-1) - 2n + \sqrt{[(n-1)^2(\sigma-1) - 2n]^2 - 4n^2} \right]^{\frac{1}{\sigma}}}. \end{aligned} \quad (\text{S2})$$

	v_1	1	1.5	2	3	4	5	6	7	8	9	10	20	50	100	∞
$\alpha = .1$	\hat{p}_1	.500	.517	.528	.542	.550	.555	.559	.562	.564	.565	.567	.573	.577	.578	.580
	\hat{w}_1	1	.743	.599	.436	.344	.285	.243	.212	.188	.169	.153	.080	.033	.017	0
$\alpha = .5$	\hat{p}_1	.500	.510	.517	.526	.531	.534	.537	0.538	.540	0.541	.542	.546	.549	.550	.551
	\hat{w}_1	1	.704	.547	0.381	.294	.239	.202	.174	.153	.137	.124	.064	.026	.013	0
$\alpha = .9$	\hat{p}_1	.500	.502	.504	.506	.507	.508	.509	.509	.509	.510	.510	.511	.511	.512	.512
	\hat{w}_1	1	.673	.508	.342	.258	.207	.173	.148	.130	.116	.104	.052	.021	.011	0

Table 1: Computed values of \hat{p}_1 and \hat{w}_1 used in Figure 5 of Lagerlöf (2019).

Similarly we can write

$$\begin{aligned}
h(n) &< \Xi_H \Leftrightarrow \left(\frac{\alpha}{1-\alpha}\right)^\sigma n^{-\sigma} < \frac{(n-1)^2(\sigma-1)-2n}{2n^2} + \frac{1}{2n} \sqrt{\frac{[(n-1)^2(\sigma-1)-2n]^2}{n^2} - 4} \Leftrightarrow \\
\left(\frac{\alpha}{1-\alpha}\right)^\sigma &< \frac{(n-1)^2(\sigma-1)-2n + \sqrt{[(n-1)^2(\sigma-1)-2n]^2 - 4n^2}}{2n^{2-\sigma}} \\
&= \frac{[(n-1)^2(\sigma-1)-2n]^2 - \left[[(n-1)^2(\sigma-1)-2n]^2 - 4n^2\right]}{2n^{2-\sigma} \left[(n-1)^2(\sigma-1)-2n - \sqrt{[(n-1)^2(\sigma-1)-2n]^2 - 4n^2}\right]} \\
&= \frac{2n^\sigma}{(n-1)^2(\sigma-1)-2n - \sqrt{[(n-1)^2(\sigma-1)-2n]^2 - 4n^2}} \Leftrightarrow \\
\frac{\alpha}{1-\alpha} &< \frac{2^{\frac{1}{\sigma}} n}{\left[(n-1)^2(\sigma-1)-2n - \sqrt{[(n-1)^2(\sigma-1)-2n]^2 - 4n^2}\right]^{\frac{1}{\sigma}}} \Leftrightarrow \\
\alpha &< \frac{2^{\frac{1}{\sigma}} n}{2^{\frac{1}{\sigma}} n + \left[(n-1)^2(\sigma-1)-2n - \sqrt{[(n-1)^2(\sigma-1)-2n]^2 - 4n^2}\right]^{\frac{1}{\sigma}}}. \quad (\text{S3})
\end{aligned}$$

The expressions in (S2) and (S3) are then evaluated at $n = 10$. The resulting expressions can then, in principle, be plotted with the help of some appropriate software. However, I have instead computed values of the right-hand sides of (S2) and (S3), evaluated at $n = 10$ and different σ 's. Then I plotted the associated pairs of (σ, α) using the \LaTeX package TikZ. \square

2.3. Calculations Used for Figure 5

Recall from the proof of Proposition 10 in Lagerlöf (2019) that \hat{p} is characterized by $F(\hat{p}_1) = 0$, where

$$F(p_1) = \frac{(1-2p_1)(r\beta+1)}{p_1(1-p_1)\left[(r\beta)^2 p_1(1-p_1) + r\beta + 1\right]} + \frac{r\beta(v_1-v_2)}{r\beta[p_1 v_1 + (1-p_1)v_2] + v_1 + v_2}.$$

Also recall that \hat{w}_1 is given by

$$\hat{w}_1 = w_2 \left(\frac{\hat{p}_1}{1-\hat{p}_1}\right)^{1+r\beta} \left(\frac{r\beta(1-\hat{p}_1)+1}{r\beta\hat{p}_1+1} \frac{v_2}{v_1}\right)^{rt}.$$

Now set $r = t = v_2 = 1$. Moreover, to start with, assume $\alpha = \beta = \frac{1}{2}$. We then get

$$F(p_1) = \frac{(1 - 2p_1) \left(\frac{1}{2} + 1\right)}{p_1(1 - p_1) \left(\left(\frac{1}{2}\right)^2 p_1(1 - p_1) + \frac{1}{2} + 1\right)} + \frac{\frac{1}{2}(v_1 - 1)}{\frac{1}{2}(p_1 v_1 + 1 - p_1) + v_1 + 1} \quad (\text{S4})$$

and

$$\hat{w}_1 = w_2 \left(\frac{\hat{p}_1}{1 - \hat{p}_1}\right)^{\frac{3}{2}} \frac{\frac{1}{2}(1 - \hat{p}_1) + 1}{\frac{1}{2}\hat{p}_1 + 1} \frac{1}{v_1} = \frac{w_2}{v_1} \left(\frac{\hat{p}_1}{1 - \hat{p}_1}\right)^{\frac{3}{2}} \frac{3 - \hat{p}_1}{2 + \hat{p}_1}. \quad (\text{S5})$$

By using Maple and the expression in (S4), the equality $F(\hat{p}_1) = 0$ can be solved for \hat{p}_1 , given various values of v_1 . Thereafter, by plugging \hat{p}_1 into (S5), we can compute \hat{w}_1 . Doing this yields the numbers in rows 3 and 4 (i.e., the ones for $\alpha = 0.5$) of Table 1 in the present document. The numbers for $\alpha = 0.1$ and $\alpha = 0.9$ are obtained similarly. \square

References

Lagerlöf, Johan N. M. 2019. "Hybrid All-Pay and Winner-Pay Contests." September, Mimeo, University of Copenhagen.