Online Appendix to
"Hybrid All-Pay and Winner-Pay Contests"

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Contents

1. Introduction

In this online appendix, I provide some proofs that were omitted from Lagerlöf (2020). In particular, I here show the calculations that were used for Figures 2, 4, and 5 of that paper.

2. Proofs of Results Not Proven in the Paper

2.1. Calculations Used for Figure 2

Assume a CES production function, a CSF of the generalized Tullock form (as in eq. (9) in Lagerlöf, 2020), and that $t = 1$ and $r \leq 1$. Under these assumptions, condition (i) in Assumption 1 is satisfied for all $\sigma \leq 1$. Thus suppose that $\sigma > 1$. Table 1 in Lagerlöf (2020) tells us that, under the stated assumptions, $\eta\left(\frac{1}{p_i}\right) = \left(\frac{\alpha}{1-\alpha}\right)^{\sigma} p_i^{\sigma-1} / \left[\left(\frac{\alpha}{1-\alpha}\right)^{\sigma} p_i^{\sigma-1} + 1\right]$. For $\sigma > 1$, this expression is strictly increasing in p_i. Therefore, since $p_i \leq 1$, an upper bound on $\eta\left(\frac{1}{p_i}\right)$ is given by $\left(\frac{\alpha}{1-\alpha}\right)^{\sigma} / \left[\left(\frac{\alpha}{1-\alpha}\right)^{\sigma} + 1\right]$. It follows that condition (i) in Assumption 1 (i.e., $r\eta\left(\frac{1}{p_i}\right)\sigma \leq 2$) is satisfied for all $p_i \in [0, 1]$ if

$$r\frac{\left(\frac{\alpha}{1-\alpha}\right)^{\sigma}}{\left(\frac{\alpha}{1-\alpha}\right)^{\sigma} + 1}\sigma \leq 2 \Leftrightarrow (r\sigma - 2)\left(\frac{\alpha}{1-\alpha}\right)^{\sigma} \leq 2.$$

This inequality is satisfied for all $\sigma \leq 2/r$. Suppose $\sigma > 2/r$. Then the inequality can be rewritten as

$$\alpha \leq \frac{\left(\frac{2}{r\sigma-2}\right)^{\frac{1}{\sigma}}}{1+\left(\frac{2}{r\sigma-2}\right)^{\frac{1}{\sigma}}} \stackrel{\text{def}}{=} \Theta\left(\sigma, r\right).$$

This is the function that is graphed in Figure 2 in Lagerlöf (2020). Note that the derivative of $\Theta\left(\sigma, r\right)$ has the same sign as the derivative of $\frac{1}{\sigma}\left[\ln 2 - \ln\left(r\sigma - 2\right)\right]$. Differentiating the latter expression with respect to σ yields

$$\frac{\ln\left(r\sigma-2\right) - \ln 2 - \frac{r\sigma}{r\sigma-2}}{\sigma^2}, \tag{S1}$$

which clearly is negative for all $r\sigma \leq 4$. Moreover, the numerator in (S1) is increasing in σ and for sufficiently large values of σ the numerator is positive. Thus, for all $\sigma \leq 4/r$, $\Theta\left(\sigma, r\right)$ is downward-sloping and there is a unique σ, such that $\sigma > 4/r$, for which $\Theta\left(\sigma, r\right)$ is minimized. This value of σ, which I denote by $\sigma = \sigma^*$, is characterized by $\ln\left(r\sigma^* - 2\right) - \ln 2 - \frac{r\sigma^*}{r\sigma^*-2} = 0$. The values of σ^* shown in the table in Figure 2 in Lagerlöf (2020) are obtained by, using Maple, solving this equation for different r values. The table also shows the associated minimized values values of $\Theta\left(\sigma, r\right)$, denoted by $\alpha^* = \Theta\left(\sigma^*, r\right)$. □

2.2. Calculations Used for Figure 4

In Figure 4 in Lagerlöf (2020) there are two graphs that indicate the part of the parameter space where R^{H} is decreasing in n (at $n = 10$). I here describe how these graphs were obtained. By assuming a CES production function (which implies $h\left(n\right) = \left(\frac{\alpha}{(1-\alpha)n}\right)^{\sigma}$) and by setting $t = r = 1$, we can write

$$\begin{aligned}
h\left(n\right) &> \Xi_L \Leftrightarrow \left(\frac{\alpha}{1-\alpha}\right)^{\sigma} n^{-\sigma} > \frac{\left(n-1\right)^2\left(\sigma-1\right)-2n}{2n^2} - \frac{1}{2n}\sqrt{\frac{\left[\left(n-1\right)^2\left(\sigma-1\right)-2n\right]^2}{n^2} - 4} \Leftrightarrow \\
\left(\frac{\alpha}{1-\alpha}\right)^{\sigma} &> \frac{\left(n-1\right)^2\left(\sigma-1\right)-2n-\sqrt{\left[\left(n-1\right)^2\left(\sigma-1\right)-2n\right]^2-4n^2}}{2n^{2-\sigma}} \\
&= \frac{\left[\left(n-1\right)^2\left(\sigma-1\right)-2n\right]^2 - \left[\left[\left(n-1\right)^2\left(\sigma-1\right)-2n\right]^2-4n^2\right]}{2n^{2-\sigma}\left[\left(n-1\right)^2\left(\sigma-1\right)-2n+\sqrt{\left[\left(n-1\right)^2\left(\sigma-1\right)-2n\right]^2-4n^2}\right]} \\
&= \frac{2n^{\sigma}}{\left(n-1\right)^2\left(\sigma-1\right)-2n+\sqrt{\left[\left(n-1\right)^2\left(\sigma-1\right)-2n\right]^2-4n^2}} \Leftrightarrow
\end{aligned}$$

$$\begin{aligned}
\frac{\alpha}{1-\alpha} &> \frac{2^{\frac{1}{\sigma}}n}{\left[\left(n-1\right)^2\left(\sigma-1\right)-2n+\sqrt{\left[\left(n-1\right)^2\left(\sigma-1\right)-2n\right]^2-4n^2}\right]^{\frac{1}{\sigma}}} \Leftrightarrow \\
\alpha &> \frac{2^{\frac{1}{\sigma}}n}{2^{\frac{1}{\sigma}}n + \left[\left(n-1\right)^2\left(\sigma-1\right)-2n+\sqrt{\left[\left(n-1\right)^2\left(\sigma-1\right)-2n\right]^2-4n^2}\right]^{\frac{1}{\sigma}}}.
\end{aligned} \tag{S2}$$

	v_1	1	1.5	2	3	4	5	6	7	8	9	10	20	50	100	∞
$\alpha = .1$	$\widehat{p}_1$	.500	.517	.528	.542	.550	.555	.559	.562	.564	.565	.567	.573	.577	.578	.580
	$\widehat{w}_1$	1	.743	.599	.436	.344	.285	.243	.212	.188	.169	.153	.080	.033	.017	0
$\alpha = .5$	$\widehat{p}_1$	.500	.510	.517	.526	.531	.534	.537	0.538	.540	0.541	.542	.546	.549	.550	.551
	$\widehat{w}_1$	1	.704	.547	0.381	.294	.239	.202	.174	.153	.137	.124	.064	.026	.013	0
$\alpha = .9$	$\widehat{p}_1$	.500	.502	.504	.506	.507	.508	.509	.509	.509	.510	.510	.511	.511	.512	.512
	$\widehat{w}_1$	1	.673	.508	.342	.258	.207	.173	.148	.130	.116	.104	.052	.021	.011	0

Table 1: Computed values of $\widehat{p}_1$ and $\widehat{w}_1$ used in Figure 5 of Lagerlöf (2020).

Similarly we can write

$$\begin{aligned}
h(n) &< \Xi_H \Leftrightarrow \left(\frac{\alpha}{1-\alpha}\right)^{\sigma} n^{-\sigma} < \frac{(n-1)^2(\sigma-1)-2n}{2n^2} + \frac{1}{2n}\sqrt{\frac{\left[(n-1)^2(\sigma-1)-2n\right]^2}{n^2} - 4} \Leftrightarrow \\
\left(\frac{\alpha}{1-\alpha}\right)^{\sigma} &< \frac{(n-1)^2(\sigma-1)-2n+\sqrt{\left[(n-1)^2(\sigma-1)-2n\right]^2-4n^2}}{2n^{2-\sigma}} \\
&= \frac{\left[(n-1)^2(\sigma-1)-2n\right]^2 - \left[\left[(n-1)^2(\sigma-1)-2n\right]^2-4n^2\right]}{2n^{2-\sigma}\left[(n-1)^2(\sigma-1)-2n-\sqrt{\left[(n-1)^2(\sigma-1)-2n\right]^2-4n^2}\right]} \\
&= \frac{2n^{\sigma}}{(n-1)^2(\sigma-1)-2n-\sqrt{\left[(n-1)^2(\sigma-1)-2n\right]^2-4n^2}} \Leftrightarrow
\end{aligned}$$

$$\begin{aligned}
\frac{\alpha}{1-\alpha} &< \frac{2^{\frac{1}{\sigma}} n}{\left[(n-1)^2(\sigma-1)-2n-\sqrt{\left[(n-1)^2(\sigma-1)-2n\right]^2-4n^2}\right]^{\frac{1}{\sigma}}} \Leftrightarrow \\
\alpha &< \frac{2^{\frac{1}{\sigma}} n}{2^{\frac{1}{\sigma}} n + \left[(n-1)^2(\sigma-1)-2n-\sqrt{\left[(n-1)^2(\sigma-1)-2n\right]^2-4n^2}\right]^{\frac{1}{\sigma}}}. \qquad \text{(S3)}
\end{aligned}$$

The expressions in (S2) and (S3) are then evaluated at $n = 10$. The resulting expressions can then, in principle, be plotted with the help of some appropriate software. However, I have instead computed values of the right-hand sides of (S2) and (S3), evaluated at $n = 10$ and different σ's. Then I plotted the associated pairs of (σ, α) using the LaTeX package TikZ. □

2.3. Calculations Used for Figure 5

Recall from the proof of Proposition 10 in Lagerlöf (2020) that $\widehat{p}$ is characterized by $\digamma(\widehat{p}_1) = 0$, where

$$\digamma(p_1) = \frac{(1-2p_1)(r\beta+1)}{p_1(1-p_1)\left[(r\beta)^2 p_1(1-p_1)+r\beta+1\right]} + \frac{r\beta(v_1-v_2)}{r\beta\left[p_1 v_1 + (1-p_1)v_2\right]+v_1+v_2}.$$

Also recall that $\widehat{w}_1$ is given by

$$\widehat{w}_1 = w_2 \left(\frac{\widehat{p}_1}{1-\widehat{p}_1} \right)^{1+r\beta} \left(\frac{r\beta\left(1-\widehat{p}_1\right)+1}{r\beta\widehat{p}_1+1} \frac{v_2}{v_1} \right)^{rt}.$$

Now set $r = t = v_2 = 1$. Moreover, to start with, assume $\alpha = \beta = \frac{1}{2}$. We then get

$$F\left(p_1\right) = \frac{\left(1-2p_1\right)\left(\frac{1}{2}+1\right)}{p_1\left(1-p_1\right)\left(\left(\frac{1}{2}\right)^2 p_1\left(1-p_1\right)+\frac{1}{2}+1\right)} + \frac{\frac{1}{2}\left(v_1-1\right)}{\frac{1}{2}\left(p_1 v_1 + 1 - p_1\right) + v_1 + 1} \tag{S4}$$

and

$$\widehat{w}_1 = w_2 \left(\frac{\widehat{p}_1}{1-\widehat{p}_1} \right)^{\frac{3}{2}} \frac{\frac{1}{2}\left(1-\widehat{p}_1\right)+1}{\frac{1}{2}\widehat{p}_1+1} \frac{1}{v_1} = \frac{w_2}{v_1} \left(\frac{\widehat{p}_1}{1-\widehat{p}_1} \right)^{\frac{3}{2}} \frac{3-\widehat{p}_1}{2+\widehat{p}_1}. \tag{S5}$$

By using Maple and the expression in (S4), the equality $F\left(\widehat{p}_1\right) = 0$ can be solved for $\widehat{p}_1$, given various values of v_1. Thereafter, by plugging $\widehat{p}_1$ into (S5), we can compute $\widehat{w}_1$. Doing this yields the numbers in rows 3 and 4 (i.e., the ones for $\alpha = 0.5$) of Table 1 in the present document. The numbers for $\alpha = 0.1$ and $\alpha = 0.9$ are obtained similarly. □

References

Lagerlöf, Johan N. M. 2020. "Hybrid All-Pay and Winner-Pay Contests." *American Economic Journal: Microeconomics*. Forthcoming.