Bertrand under Uncertainty: Private and Common Costs

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Abstract

Does private information about costs in a homogeneous-good Bertrand model soften competition? Earlier literature has shown that the answer (perhaps counter-intuitively) is "no," while assuming (i) private (i.e., independent) cost draws and (ii) no drastic innovations. I first show, in a fairly general setting, that by relaxing (i) and instead allowing for sufficiently much common (interdependent) cost draws, private information indeed softens competition. I then study a specification that yields a closed-form solution and show that relaxing (ii) but not (i) does not alter the result in the earlier literature. While relying on specific functional forms, this specification is quite rich and might be useful in applications. It allows for any (positive) degree of interdependence between the cost draws, for any demand elasticity, and for any number of firms. The closed-form solution is simple and in pure strategies.

Keywords: Bertrand competition, Hansen-Spulber model, private information, information sharing, common values, private values

JEL classification: D43 (Oligopoly and Other Forms of Market Imperfection), D44 (Auctions), L13 (Oligopoly and Other Imperfect Markets)

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1 Introduction

The Bertrand model of price competition predicts that price equals marginal cost and that firms earn zero profit—a result which is often referred to as the Bertrand paradox, as it suggests that the presence of only two firms is sufficient to eliminate all market power and give rise to the perfectly competitive outcome. The paradox has prompted scholars to study various extensions and variations of Bertrand's original model, thereby identifying several model features that, if added to the standard setup, resolve the paradox by providing some amount of market power to the firms. One such variation is to assume that each price-setting firm has private information about a characteristic of its production technology, for example the firm's (constant) marginal cost. This is an assumption that is empirically very plausible, and it yields an equilibrium price that is strictly above the marginal cost.[1] Moreover, the equilibrium price, and thus each firm's amount of market power, are decreasing in the number of firms in the market. Hence, this modification of the standard Bertrand setup, which I will call the *Hansen-Spulber model*,[2] yields predictions that are qualitatively identical to those of the classic Cournot model, while avoiding that model's unappealing feature that no firm or individual sets the price.

The Hansen-Spulber model can thus, potentially, be an important analytic tool for studying oligopoly markets. However, this requires that we have a good understanding of the tool and its logic. In particular, is it indeed private information/uncertainty that creates market power, as one might be tempted to conclude from the above text? More generally, what is the effect of cost uncertainty in the Bertrand model on profits, consumer surplus, and welfare? The present paper investigates those questions in a setting in which firms' cost parameters are interdependent. Previous literature has studied the case where costs are independent. The case with interdependent costs is important for two reasons. First, it is empirically highly relevant. We would expect that, typically, all firms in a market are affected by external and economy-wide shocks to wages, interest rates, the price of material and energy, etc. Second, the extent to which costs are independent/interdependent is crucial for the results.

I study a static, simultaneous-move oligopoly model with n price-setting firms that sell a homogeneous good and face a downward-sloping demand function.[3] The firms' marginal costs are constant, and each firm draws a signal about the own cost parameter's value from a common distribution. Although the *signals* are independent, the way in which they are related to a firm's cost parameter makes the *costs* potentially interdependent: by varying a parameter, the analysis can capture all cases from independent (or private) costs to perfectly interdependent (or common) costs. I compare the equilibrium behavior in this model with the equilibrium behavior in a model that is identical to the original one except that all cost draws are common knowledge. The objective is to learn whether private information softens or intensifies price competition.

Earlier literature, in particular Hansen (1988), has shown that, for the case with independent cost draws, downward-sloping demand, and a duopoly, private information *intensifies* price

[1] The firm that draws the lowest marginal cost (and thus charges the lowest price) earns a positive profit ex post. The other firms earn a zero profit ex post, but their expected profit, at the stage before they have learned their cost parameter, is positive.

[2] It was first studied by Hansen (1988) and Spulber (1995).

[3] The analysis also covers the case with perfectly inelastic demand.

competition. That may sound surprising. To understand how this result is compatible with the claims in the text above, note that when we assume private information about each firm's own cost parameter, we implicitly also introduce the possibility that these parameters differ from each other. It is well-known (e.g., from IO textbooks) that cost heterogeneity, in an otherwise standard Bertrand setting, creates market power. Hansen's analysis says that if we add private information to the setup with potentially different but commonly known costs, then the expected market price drops. In that sense, private information as such intensifies competition; the market power that is present in the Hansen-Spulber model with independent cost draws stems exclusively from cost heterogeneity.

It is instructive to look at what broad arguments Hansen (1988) uses when proving the result that—given independent cost draws—the Bertrand model with private information yields a lower expected price than the Bertrand model with complete information. Hansen's model is couched in auction theory terms. Thus, private and complete information correspond to a (first-price) *sealed*-bid and an *open*-bid auction, respectively; moreover, a Bertrand model with downward-sloping and with perfectly inelastic demand correspond to an auction with *variable* and *fixed* quantity, respectively. With this terminology, the result that Hansen proves is that $\mathbb{E}\left[p \mid \text{open, variable}\right] > \mathbb{E}\left[p \mid \text{sealed, variable}\right]$, where p is the market price. Hansen first notes that, in an open auction, the equilibrium pricing strategies are the same regardless of whether the quantity is variable or fixed (or, in oligopoly language, whether demand is downward-sloping or perfectly inelastic). In either case, the lowest-cost firm can win the whole market with a price that equals the marginal cost of the firm with the second-lowest cost. This yields equality (a) in (1):

$$\mathbb{E}\left[p \mid \text{open, variable}\right] \overset{(a)}{=} \mathbb{E}\left[p \mid \text{open, fixed}\right] \overset{(b)}{=} \mathbb{E}\left[p \mid \text{sealed, fixed}\right] \overset{(c)}{>} \mathbb{E}\left[p \mid \text{sealed, variable}\right]. \tag{1}$$

The argument that yields equality (a) relies critically on Hansen's assumption that the lowest-cost firm's optimal monopoly price always exceeds the marginal cost of the firm with the second-lowest cost (i.e., a "drastic innovation" must not be possible); without that assumption, the equality would be replaced by a "$<$"-sign and Hansen's proof would no longer be valid.

Next, Hansen invokes the revenue equivalence theorem, which in his model says that in a fixed-quantity auction the expected revenue (which equals the expected price) is the same regardless of whether the auction is sealed-bid or open-bid. This is equality (b) in (1). Finally Hansen shows that, in a sealed-bid auction, the equilibrium price must be lower when the quantity is variable compared to when it is fixed (inequality (c) in (1)). The intuition for this result is straightforward. In the sealed-bid auction, if the firm raises its price, it will have a higher profit if it still wins the market, but the probability of winning has decreased. The optimal price balances those two effects. However, the former (positive) effect is smaller when demand is downward-sloping, as the higher price then leads to a loss of sales. Therefore the expected price must be lower when demand is not perfectly inelastic.

Jointly, the three steps (a), (b), and (c) yield the desired result that the expected price in the sealed-bid auction with a variable quantity is lower than the expected price in the open-bid auction with a variable quantity—or, in other words, cost uncertainty in the Bertrand model intensifies competition in the sense that it lowers the expected market price. While step (c) appears to be quite robust, it has already been noted that step (a) relies critically on Hansen's assumption about the support of the cost distribution from which the firms draw their costs. It is also clear from the above reasoning that Hansen's proof relies on the revenue equivalence theorem (step (b)). In an environment where that theorem does not hold, the proof will not be valid and it is again conceivable that the result could be reversed. One such environment is a common value auction (or, in oligopoly language, a model with common costs). It is known that in such an auction we have $\mathbb{E}[p \mid \text{open, fixed}] < \mathbb{E}[p \mid \text{sealed, fixed}]$;[4] that is, equality (b) is replaced by a "$<$"-sign. Whether Hansen's result is reversed would then depend on the relative magnitudes of the inequalities (b) and (c), as they point in different directions.

The present paper studies the effect of relaxing each one of the assumptions of independent cost draws (which matters for step (b)) and of no drastic innovations (which step (a) above relies on). To start with, Proposition 2 reports results for the case with perfectly inelastic demand but an arbitrary amount of interdependence between the cost draws. As long as the cost draws are not perfectly independent, the expected price is higher and the expected consumer surplus is lower with incomplete information—suggesting that private information is anti-competitive. Moreover, under the same condition, expected industry profits are higher with incomplete information, which again can be thought of as an expression of more market power. Next, Proposition 3 does allow for downward-sloping demand. It states that, also in this environment, the three results mentioned above about expected price, consumer surplus, and industry profits hold if the cost draws are sufficiently much common.

The results reported in Propositions 2 and 3 thus overturn the results in Hansen (1988)—with sufficiently common costs, private information relaxes competition. Intuitively, we can understand this result in terms of the so-called winner's curse, discussed in the auction literature. Learning that one has won the market (i.e., chosen the lowest price) might be bad news, as it suggests that the own estimate of the (partially) common cost is too optimistic. To protect itself from this possibility, a firm should compute the expected value of the cost contingent on winning, which leads to a higher estimate. The higher cost estimate in turn leads to a higher optimally chosen price. This helps explain our result that, with common costs, private information relaxes competition. (See Section 3.3 for further discussion, and a caveat.)

The results in Propositions 2 and 3 are valid in a fairly general setting. To obtain stronger results, I next assume a more specific setup that yields a closed-form solution. Although this specification relies on specific functional forms, it is sufficiently rich to allow for any given number of firms, any level of demand elasticity, and any (positive) level of interdependence between the firms' costs—private costs, common costs, and combinations of those extremes. In addition, the model allows for many different values of the firms' ex ante efficiency level (i.e., a parameter that measures their common likelihood of drawing a relatively low cost signal). The closed-form solution is simple, in pure strategies, and involves price dispersion.[5] Given

[4] See Milgrom and Weber (1982) and Krishna (2002, Ch. 6).

[5] Wolfstetter (1999), Lofaro (2002), Abbink and Brandts (2007), and Belleflamme and Peitz (2015) all derive closed-form solutions in a version of the Hansen-Spulber model with a uniform cost distribution on $[0, 1]$ and linear demand. (In their experimental study, Abbink and Brandts (2007) assume a uniform distribution on $[0, 99]$;

this model specification, Proposition 4 first states a sufficient condition and then, for a duopoly market, a necessary and sufficient condition under which private information yields a lower expected price than complete information.

Finally, and still using the setup that yields a closed-form solution, Proposition 5 assumes independent cost draws and downward-sloping demand. The proposition states that, under those assumptions, the expected price is lower with private information than with complete information. The implication of this result is that, at least in the setting under consideration, Hansen's assumption about the cost distribution—implying that the lowest-cost firm's optimal monopoly price always exceeds the second-lowest cost draw—does not matter for his results. The result in Proposition 5 also holds for any arbitrary number of firms, whereas Hansen assumed a duopoly market. We can conclude that, both in Hansen's (1988) model and in the present environment with private costs, asymmetric information is not anti-competitive but pro-competitive, at least in the sense that the firms' equilibrium mark-ups are (in expectation) smaller in an environment with asymmetric information.

The Bertrand model with private information about cost has also been studied by Spulber (1995). Although Spulber's study was published after Hansen's (1988), it is apparently carried out independently of his; moreover, Spulber's model is more general than Hansen's. In contrast to Hansen, who assumes a duopoly, Spulber allows for an arbitrary number of firms. Furthermore, Spulber's analysis is consistent with private information about any characteristic of the firm's production technology (not necessarily a marginal cost parameter and not relying on constant returns to scale).[6] Spulber shows that, in that setting, there is a unique and symmetric equilibrium price, which is increasing in the own marginal cost. Importantly, the equilibrium price lies strictly between the marginal cost and the monopoly price, which means that the firms have some market power and earn a positive profit. Spulber does not make the comparison between the private- and complete-information models in terms of competitiveness. However, he conjectures that, in the setting he studies, private information softens competition (Spulber, 1995, p. 10, emphasis added):

> Asymmetric information thus plays an important role in imperfect competition. In [the model] studied here, the surgical precision that is required to price slightly below [...] higher cost rivals is eliminated by the lack of exact knowledge about the characteristics of the rivals. In the short run, with market structure fixed, *asymmetric information appears to reduce competition*[...].

It is not only Spulber himself who interprets his results in this fashion, so do several other

but this specification is equivalent to the one in the other three papers/books, although with another scaling of the units in which output and cost are measured.) Wolfstetter (1999) studies a version of the model with a uniform cost distribution on $[0, 1]$ and perfectly inelastic demand and also derives a closed-form solution (this is effectively a standard private-value first-price auction model). The specification yielding a closed-form solution used in the present paper is a substantial generalization of all the above models.

[6] The model that Spulber and Hansen study has also found its way into textbooks—see Wolfstetter (1999, pp. 236-37) and Belleflamme and Peitz (2015, pp. 47-49). Arozamena and Weinschelbaum (2009) study a sequential version of Spulber's model and compare with the simultaneous-move version. Lofaro (2002) obtains a closed-form solution of Spulber's model by assuming a uniform distribution, and he then compares the price competition outcomes with the quantity competition outcomes. Athey (2002, p. 198) generalizes some of Spulber's results in a number of directions, allowing for, among other things, asymmetric cost distributions. Abbink and Brandts (2007) test Spulber's model in the laboratory.

authors. For example, Spiegel and Tookes (2008, p. 33, fn. 33) write that "Spulber (1995) also shows how, in Bertrand competition, not knowing rivals' costs implies equilibrium prices that are above marginal costs (i.e., information asymmetry softens product market competition)."[7] Therefore, an additional contribution of the present paper is to serve as a reminder that, in the setting assumed by both Hansen (1988) and Spulber (1995), private information is indeed pro-competitive.

There is a related literature that studies firms' incentive to share information about their own marginal cost parameter under Bertrand competition with *differentiated* goods. Gal-Or (1986) analyzes such a model with two firms and independent cost draws. She shows that not to share information is a dominant strategy for each duopolist. Raith (1996) considers the more general case with n firms and does allow for correlated cost draws. Also Raith, however, assumes differentiated goods. Moreover, neither Gal-Or (1986) nor Raith (1996) makes any comparison between market outcomes with complete and incomplete information. Other related literature includes Milgrom and Weber's (1982) seminal analysis of auctions with affiliated values. Most of their paper concerns the case where bidders are risk neutral and can purchase one unit of a good. However, Section 8 of their paper also considers the case with risk averse bidders, which is similar in spirit to a model with downward-sloping demand. Milgrom and Weber show that, "for models that include both affiliation and risk aversion, the first- and second-price auctions [which, in my setting, correspond to incomplete and complete information] cannot generally be ranked by their expected prices" (p. 1114). The present paper assumes a particular oligopoly setting and then compares the two expected prices, for various values of the parameter that measures cost interdependence. It also makes comparisons of other entities, such as consumer surplus (which has no natural equivalent in the Milgrom-Weber model), across the two informational regimes.

2 The Model

There are $n \geq 2$ risk neutral and profit-maximizing firms that compete à la Bertrand in a homogeneous-good product market. The firms are ex ante identical, they choose their prices simultaneously, and they interact only once. Market demand is given by $D(p)$, where p denotes price. This function is twice continuously differentiable and weakly decreasing (i.e., $D'(p) \leq 0$ for all p).[8] Assume that $D(1) \geq 0$, where 1 is highest possible cost realization (see below). The firm that charges the lowest price, denoted by $p_{\min}$, sells the quantity $D(p_{\min})$, while the rival firms sell nothing and make a zero profit. If two or more firms have chosen the same price, they share the market equally.

Each firm has a production technology that is characterized by constant returns to scale,

[7] See also Wolfstetter (1999), who presents two simple versions of the model, one with perfectly inelastic and one with downward-sloping demand. He introduces the analysis of these models by stating that (p. 236): "A much simpler [relative to the model with capacity constraints] resolution of the Bertrand paradox can be found by introducing incomplete information." Yet another paper that refers (twice—on p. 638 and p. 646) to Spulber's (1995) result as a "resolution of the Bertrand paradox" is Abbink and Brandts (2007).

[8] Specifically, some parts of the analysis will assume $D'(p) < 0$, whereas other parts will assume a perfectly inelastic demand function.

with firm i's marginal cost being denoted by c_i. In particular,

$$c_i = (1-\alpha)\, s_i + \frac{\alpha}{n} \sum_{j=1}^{n} s_j, \tag{2}$$

where $\alpha \in [0,1]$ is a parameter and s_i is firm i's *signal*. In one version of the model to be studied, s_i is private information of firm i. The firms' signals are independently drawn from the cumulative distribution function $F(s_i)$, with support on the unit interval $[0,1]$; this function is twice continuously differentiable and satisfies $F'(s_i) \stackrel{\text{def}}{=} f(s_i) > 0$ for all $s_i \in (0,1)$. These assumptions imply that, for $\alpha = 0$, the firms' cost parameters are independent; this is the standard framework with *private costs* that is used in Hansen (1988) and Spulber (1995). For $\alpha = 1$ the cost parameters are identical; this is the other polar case: a framework with *common costs*. Values of α between those two extremes capture intermediate situations where the costs are not (fully) common but still interdependent. This way of modeling (partial) interdependence of values (here: costs) has, because of its tractability, often been used in the auction literature; see, e.g., Vincent (1995) or Klemperer (2004, p. 53-55). Given the above notation and assumptions, we can write the profit of a monopolist as $\pi^m(p, c_i) = (p - c_i)D(p)$. Assume that this profit function is strictly quasiconcave in p and let $p^m(c_i)$ denote the optimal monopoly price (i.e., $p^m(c_i) \in \arg\max_p \pi^m(p, c_i)$).

The specification in (2) is a simple way of capturing the ideas that there is both a private and a common component to firms' costs, and that each firm has some private information about the cost. However, the specification also, as a by-product, implies two other features. First, the formulation in (2) means that a firm, when choosing its price, does not know its own cost. Second, the specification implies that firm i's rivals know things about firm i's cost that it does not know itself. Although such situations may appear odd, they naturally arise in markets where firms, for various reasons, produce and deliver their products or services a significant amount of time after the consumers have made their purchases. An example of this is the market for direct-to-consumer DNA tests.[9] Another example would be newspapers and other companies where the customer purchases a subscription, after which the company delivers the product regularly during an extended period of time (perhaps a year). In such situations, the companies must post their prices prior to knowing about future events that could affect the costs of material, labor, energy, office rents, etc. (examples of such hard-to-predict events could be labor strikes, conflicts leading to oil price fluctuations, changes in tax rates, or pandemics). Any information about such events that is available is often dispersed among the firms in the market, as different firms tend to get their information from different sources. This characteristic of the market is captured by the model feature that firm i's cost depends on the rivals' signals.

I will solve, and then compare, two versions of the model: one where the signals s_i (and thus also the firms' cost parameters) are common knowledge and one where each signal s_i is firm i's private information.

[9]The testing company must after the purchase mail the test kit to the consumer (perhaps inter-continentally), after which a saliva sample or swab is returned to the company. The sample is then sequenced in a lab that could be capacity constrained, and using a process that involves careful quality checks. For advanced tests, several months of waiting time is common.

3 Analysis

3.1 Incomplete Information

First consider the incomplete information model. That is, assume that each signal s_i is private information of firm i, although it is common knowledge that c_i is given by (2) and that the s_i values are independently drawn from the distribution $F(s_i)$. Suppose there is a symmetric equilibrium strategy $p^I(s)$ that is strictly increasing and differentiable in the signal s that the firm observes (the superscript I stands for incomplete information). Denote the inverse of this function by $\chi(p)$, meaning that χ is the signal value that would give rise to the price p. If firm i expects all its rivals to follow the equilibrium strategy $p^I(\cdot)$, and it drew a signal s_i and chose a price p_i, then its expected profit is

$$\mathbb{E}[\pi_i \mid s_i] = \left[p_i - \widehat{\Phi}\left(\chi\left(p_i\right), s_i\right)\right] D\left(p_i\right) G\left[\chi\left(p_i\right)\right]. \tag{3}$$

The term $\widehat{\Phi}\left[\chi\left(p_i\right), s_i\right]$ is firm i's expected cost, conditional on the own signal s_i and on being the cheapest firm in the market. In particular, we can compute[10]

$$\widehat{\Phi}\left[\chi\left(p_i\right), s_i\right] \stackrel{\text{def}}{=} \mathbb{E}[c_i \mid s_i, p_i < p_j \text{ all } j \neq i] = s_i + \frac{\alpha(n-1)}{n}\left[\frac{\int_{\chi(p_i)}^{1} z f(z) dz}{1 - F\left[\chi\left(p_i\right)\right]} - s_i\right]. \tag{4}$$

The last term in (3), $G\left[\chi\left(p_i\right)\right]$, is the probability with which firm i is indeed the cheapest firm in the market and all the rivals thus having drawn signals that exceed $\chi\left(p_i\right)$:

$$G\left[\chi\left(p_i\right)\right] \stackrel{\text{def}}{=} \left[1 - F\left(\chi\left(p_i\right)\right)\right]^{n-1}. \tag{5}$$

When firm i chooses p_i to maximize (3), it trades off its desire to charge a low price in order to win the market against its desire to set a relatively high price that ensures that it earns a large profit in case it does win the market. At the optimum, the first-order condition $\partial\mathbb{E}[\pi_i \mid s_i]/\partial p_i = 0$ must hold. By rewriting this and by imposing symmetry, we obtain the following differential equation:

$$\frac{\partial p^I(s)}{\partial s} = \frac{\left[(n-1)\, p^I(s) - s - (n-2)\, \Phi\left(s\right)\right] h\left(s\right) D\left(p^I\right)}{D\left(p^I\right) + \left[p^I(s) - \Phi\left(s\right)\right] D'\left(p^I\right)}, \tag{6}$$

where $\Phi\left(s\right) \stackrel{\text{def}}{=} \widehat{\Phi}(s, s)$ and where $h\left(s\right)$ is the hazard rate associated with the signal distribution, $h\left(s\right) \stackrel{\text{def}}{=} f\left(s\right) / \left[1 - F\left(s\right)\right]$.

Proposition 1. ***(Equilibrium)*** *There is an equilibrium of the incomplete information model where each firm chooses the price $p^I(s)$ that is defined by the differential equation (6) and the boundary condition $p^I(1) = 1$. For all $s \in [0, 1)$, this equilibrium satisfies $p^I(s) > \Phi\left(s\right)$ and $\mathbb{E}[\pi_i \mid s] > 0$.*

It follows from the last claim in Proposition 1 that, in the equilibrium under consideration,

[10] The proofs of the relationship in (4) and other results that are not shown in the main text can be found in the appendix.

a firm earns a strictly positive expected profit also from an ex ante point of view (i.e., in expectation prior to having learned its own signal).[11] Intuitively, why do the firms have market power in this environment? As we learned from the discussion in the introduction of Hansen's (1988) analysis of the private-cost case, the (or at least one) reason for this is that the firms' cost realizations are different, and this enables a firm with a relatively low cost to charge a price that that cannot be profitably undercut by the rivals. This logic works in spite of the fact that (a component of) each firm's costs is private information.

3.2 Complete Information

Consider now a Bertrand model that is identical to the one above, except that the signals $(s_1, \ldots, s_n)$, and thus also the marginal cost parameters of all the firms, are common knowledge. This is a model that is often analyzed in textbooks; as already discussed in Section 2, the equilibrium outcome is that the firm with the lowest cost draw serves the whole market and charges a price that equals either the second most efficient firm's marginal cost or, if that is lower, the optimal monopoly price:

$$p^C = \min \left\{ c_{(2)}, p^m \left[c_{(1)} \right] \right\}. \tag{7}$$

Here, the superscript C is short for complete information, and the subscript (j) indicates the jth order statistic (i.e., the jth lowest realization among the n draws). In general, in this model, the optimal monopoly price may indeed be lower than the nearest rival's marginal cost. However, this cannot happen if demand is perfectly inelastic:

$$D(p) = \begin{cases} 1 & \text{if } p \leq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

3.3 Comparison

In this subsection I compare the equilibrium outcomes of the models in subsections 3.1 and 3.2 with regard to market price, consumer surplus, profits, and total surplus. The comparisons are made from an ex ante point of view, that is, in expected terms at the stage where the firms have observed neither their own nor any of their rivals' signals.

Consider first the relatively straightforward case where demand is perfectly inelastic. The results for this case are not novel, as they (at least for all intents and purposes) follow from the analysis in Milgrom and Weber (1982). It is nevertheless useful to establish these results in the present framework, before studying the case with downward-sloping demand. Let the subscript (j, m) denote the jth order statistic among m draws, with $m \leq n$ (i.e., we have $s_{(j,n)} = s_{(j)}$).

Proposition 2. ***(Perfectly inelastic demand)*** *Suppose demand is perfectly inelastic (i.e.,*

[11]The proof of Proposition 1, which can be found in the appendix, is based on the proof of Theorem 2 in Maskin and Riley (1984). In the setting of these authors, the equilibrium is unique. However, Maskin and Riley's uniqueness proof does not obviously carry over to the present setting with a (partially) common cost, and no uniqueness claim is made here. Yet, my conjecture is that, also in the present model, the equilibrium is unique.

$D(p)$ is given by (8)). Then $\mathbb{E}\left[p^C\right] = (1-\alpha\,)\,\mathbb{E}\left[s_{(2)}\right] + \alpha\,\mathbb{E}\left[s\right]$ *and*

$$\mathbb{E}\left[p^I\right] = \mathbb{E}\left[p^C\right] + \frac{\alpha}{n-1}\left[\mathbb{E}\left[s_{(2)}\right] - \mathbb{E}\left[s_{(1,n-1)}\right]\right] = \mathbb{E}\left[p^C\right] + \frac{n\alpha}{n-1}\int_0^1 F(s)\left[1-F(s)\right]^{n-1} ds. \tag{9}$$

Moreover, with incomplete instead of complete information and for all $\alpha \in [0,1]$*:*

(i) *the expected price is weakly higher (*$\mathbb{E}\left[p^I\right] \geq \mathbb{E}\left[p^C\right]$*);*

(ii) *the expected consumer surplus is weakly smaller (*$\mathbb{E}\left[S^I\right] \leq \mathbb{E}\left[S^C\right]$*);*

(iii) *the expected industry profits are weakly higher (*$\mathbb{E}\left[\Pi^I\right] \geq \mathbb{E}\left[\Pi^C\right]$*);*

(iv) *and the expected total surplus is the same (*$\mathbb{E}\left[W^I\right] = \mathbb{E}\left[W^C\right]$*).*

The inequalities in (i)-(iii) hold strictly if, and only if, $\alpha \in (0,1]$.

Proposition 2 says that, with private costs, the expected price is the same with complete and with incomplete information (this is the revenue equivalence theorem). However, for any degree of common or partially common costs (so for any $\alpha > 0$), the expected price under incomplete information is strictly higher. That is, in line with Spulber's (1995) intuition as quoted in the Introduction, uncertainty here softens competition. Proposition 2 also says that the consumer is worse off and the firm is better off under incomplete information (with indifference for $\alpha = 0$). This is simply because, with perfectly inelastic demand, expected consumer surplus and profits are linear and, respectively, decreasing and increasing in the expected price. Finally, welfare is always the same with complete and with incomplete information. The reason is that the sum of expected consumer surplus and profits does not depend on the expected price, as this cancels out.

What is the intuition for the result that, with (partially) common costs, uncertainty softens competition? We can get some help in understanding this by recalling the logic of the so-called winner's curse, discussed in the auction literature. In a common-value auction (where, say, some mineral rights are for sale), learning that the own bid was the highest is, in a sense, bad news for a bidder—for this information implies that all other bidders' estimates of the value of the mineral rights are lower than the own estimate, making the latter likely to be too optimistic. A failure to foresee this effect, and take it fully into account when choosing one's bid, will result in the winner's curse. The firms in our model do not suffer from the curse, as they compute the expected value of the own cost conditional on winning (see (4)). This makes their expected cost higher than otherwise; thus, also the firms' chosen price is higher than it would have been if the firms did not have to protect themselves from the winner's curse. It is only with at least partially common costs that this effect matters; moreover, the effect obviously does not arise when there is complete information about the cost draws. Hence, we can think of the need and desire to protect oneself from the winner's curse as an explanation for why uncertainty softens competition with (partially) common costs but not with private costs. A caveat, however, is that a firm that is naive about its relevant expected cost, by not conditioning on winning, does not choose a price that, in expectation and in general, is *exactly* the same as the one chosen with complete information. Still, naivete unambiguously lowers the price under incomplete information.[12]

Now assume that demand is indeed downward-sloping ($D'(p) < 0$). For this case and with private costs ($\alpha = 0$), Hansen's (1988) analysis told us that private information is pro-competitive ($\mathbb{E}\left[p^I\right] < \mathbb{E}\left[p^C\right]$). The next proposition states that this result is overturned if costs are sufficiently common.

Proposition 3. ***(Sufficiently common cost)*** *Assume that $D'(p) < 0$ for all p. There exists an $\widehat{\alpha} \in [0,1)$ such that, for all $\alpha \in (\widehat{\alpha}, 1]$, with incomplete instead of complete information:*

(i) *the expected price is strictly higher ($\mathbb{E}\left[p^I\right] > \mathbb{E}\left[p^C\right]$);*

(ii) *the expected consumer surplus is strictly smaller ($\mathbb{E}\left[S^I\right] < \mathbb{E}\left[S^C\right]$); and*

(iii) *the expected industry profits are strictly higher ($\mathbb{E}\left[\Pi^I\right] > \mathbb{E}\left[\Pi^C\right]$).*

3.4 Functional Forms that Yield a Closed-Form Solution

To obtain more specific results, I here consider a parameterized version of the model that allows for a closed-form solution. Thus, let the demand function be given by $D(p) = (1-p)^\epsilon$, where $\epsilon \geq 0$ is an exogenous parameter. This specification implies that, for a given price p, the parameter ϵ is proportional to the price elasticity of demand: the larger is ϵ, the more elastic is demand (and for $\epsilon = 0$, demand is perfectly inelastic).[13] Moreover, let the signal distribution be given by

$$F(s_i) = 1 - (1 - s_i)^x, \tag{10}$$

with $x > 0$ being an efficiency parameter: an increase in x makes the firms more efficient from an ex ante perspective, in the sense that this makes lower signal values more likely.[14] I will refer to the version of the model with these two functional forms as the "parameterized version of the model."

[12] To be more precise, suppose the firms do not, as they do in (4), condition on winning when computing their expected cost; that is, this expected cost is given by

$$\widehat{\Phi}_N[s_i] \stackrel{\text{def}}{=} \mathbb{E}[c_i \mid s_i] = s_i + \frac{\alpha(n-1)}{n}\left[\mathbb{E}[s] - s_i\right],$$

where the subscript N is short for naive. This means that the expected cost is not a function of the own price, which modifies the firm's first-order condition relative to the original model. Solving for the symmetric price that is consistent with that modified first-order condition, and then taking expectations, yield the following expected price:

$$\mathbb{E}\left[p_N^I\right] = \mathbb{E}\left[p^C\right] + \frac{\alpha}{n}\left[\mathbb{E}\left[s_{(2)}\right] - \mathbb{E}[s]\right].$$

This equals $\mathbb{E}\left[p^C\right]$ exactly only for particular signal distributions and for certain values of n (it can be either lower or higher). Still, we always have $\mathbb{E}\left[p_N^I\right] < \mathbb{E}\left[p^I\right]$.

[13] In particular, the price elasticity of demand equals $\eta(p) \stackrel{\text{def}}{=} -D'(p)p/D(p) = \epsilon p/(1-p)$.

[14] We can think of each firm i in the market as having access to x research laboratories, each of which produces one independent signal draw from a uniform distribution on $[0,1]$; the firm then uses the lowest one of the x draws as its signal. This procedure is equivalent to receiving one single draw from the distribution $F(s_i)$, defined in (10) above. Obviously, however, the analysis does not rely on this interpretation or on x being an integer.

3.4.1 Incomplete Information

First consider the parameterized version of the model under incomplete information.

Corollary 1. ***(Closed-form solution)*** *The following is an equilibrium strategy of the parameterized version of the model:*

$$p^I(s_i) = A + (1-A)\, s_i, \qquad \textit{where} \qquad A \overset{def}{=} \frac{n(1-\alpha)+\alpha}{n[1+\epsilon+(n-1)x]} + \frac{\alpha(n-1)}{n(1+x)}. \tag{11}$$

This version of the model thus has an equilibrium that is quite simple, with each firm charging a price that is linear (or affine) in its own signal.[15] The linear equilibrium strategy satisfies $p^I(1) = 1$, and for all $s_i < 1$ the firm chooses a price strictly above its expected cost conditional on winning, $\Phi(s_i)$. Indeed, in the parameterized version of the model, we can write

$$\Phi(s_i) = C + (1-C)\, s_i, \qquad \text{with} \qquad C \overset{\text{def}}{=} \frac{\alpha(n-1)}{n(1+x)}, \quad A > C, \quad \text{and} \quad \lim_{\epsilon\to\infty} A = C. \tag{12}$$

The equilibrium pricing strategy in (11) features all the comparative statics properties that we would expect in this economic environment: the pricing schedule's distance to the expected cost schedule (i.e., $A - C$) is strictly decreasing in the number of firms (n), in the elasticity parameter (ϵ), and in the firms' common level of ex ante efficiency (x). This is reassuring, as it suggests that even if the parameterized version of the model relies on functional forms that are special, in qualitative terms it behaves in the way we would expect a more general model to do. Hence, hopefully also the results to be derived within this framework would hold more generally.[16]

3.4.2 Complete Information

Next consider the parameterized version of the model under complete information. The optimal price p^C of the firm that wins the market is, again, given by (7), although now we can write

$$p^m\left[c_{(1)}\right] = \frac{1+\varepsilon c_{(1)}}{1+\varepsilon}. \tag{13}$$

The optimal monopoly price in (13) is guaranteed to exceed the second lowest cost parameter (meaning that a drastic innovation is not possible) only if the costs are common to a high enough degree. In particular, we have:

[15] As mentioned in the Introduction, this result is a generalization of the models in the existing literature that yield a closed-form solution. By setting $(\alpha, \epsilon, x) = (0, 1, 1)$, we obtain the specifications used in Lofaro (2002), Belleflamme and Peitz (2015), and the downward-sloping-demand model in Wolfstetter (1999, p. 237). By setting $(\alpha, \epsilon, x) = (0, 0, 1)$, we obtain the specification in Wolfstetter's (1999, p. 236) model with perfectly inelastic demand.

[16] It is not so clear what comparative statics result to expect in terms of a parameter like α, which captures the strength of the common cost feature (it should depend on the particular specification used to model private/common costs). In the framework that is used here, the equilibrium pricing schedule's distance to the expected cost schedule is strictly decreasing in α: moving closer to fully common costs leads to a smaller price-cost margin.

Lemma 1. ***(Possibility of drastic innovations)*** *The relationship* $c_{(2)} \leq p^m \left[c_{(1)}\right]$ *holds for all signal realizations if, and only if,*

$$\alpha \geq \frac{n\varepsilon}{1+n\varepsilon}. \tag{14}$$

3.4.3 Comparison

Comparing the two versions of the model, we obtain the following result.

Proposition 4. *Consider the parameterized version of the model. Assume* $\varepsilon > 0$.

- ***(a) (Sufficient condition)*** *For all* α *that satisfy (14) and for all* $n \geq 2$*, the expected price is strictly higher with incomplete instead of complete information* ($\mathbb{E}[p^I] > \mathbb{E}[p^C]$)*.*

- ***(b) (Necessary and sufficient condition)*** *Assume* $n = 2$*. Then there exists a unique* α *such that* $\mathbb{E}[p^I] = \mathbb{E}[p^C]$*. This value of* α*, denoted by* α^**, satisfies*

$$\alpha^* \in \left(0, \frac{2\varepsilon}{1+2\varepsilon}\right) \qquad \text{and} \qquad \lim_{\epsilon \to 0} \alpha^* = 0.$$

Moreover, we have $\mathbb{E}[p^I] < \mathbb{E}[p^C]$ *for all* $\alpha \in [0, \alpha^*)$*, and* $\mathbb{E}[p^I] > \mathbb{E}[p^C]$ *for all* $\alpha \in (\alpha^*, 1]$*.*

Thus, Proposition 4 tells us that private information softens competition if costs are sufficiently much common, and that it strengthens competition otherwise (at least for $n = 2$). We also know, from Proposition 2, that, if $\epsilon = 0$, private information softens competition for all $\alpha > 0$. In qualitative terms, those results are summarized in Figure 1, which depicts the (ϵ, α)-space. North-west in that space, private information softens competition; and in the south-east region, private information strengthens competition. Intuitively, the results can be understood with the help of the series of (in-) equality signs in (1) stated in the Introduction, together with the discussion about the winner's curse that follows Proposition 2. Step (c) in (1) tends to make private information conducive to *tougher* competition; moreover, this effect is stronger, the larger is ϵ (so eastwards in Figure 1). The logic behind step (c) is that, with incomplete information, a firm's optimal price is lower if demand is more elastic (with complete information, in contrast, the elasticity does not matter). Step (b) in (1) will, if costs are at least partially common, tend to make private information conducive to *softer* competition—an effect that is stronger, the larger is α (so northwards in Figure 1). We can understand this effect by noting that, with private information and (partially) common costs, the firms need to protect themselves from the winner's curse by raising their price.

As argued in the Introduction, the model studied in the present paper could conceivably overturn Hansen's (1988) results also if costs are private (i.e., with $\alpha = 0$). (If this were the case, the reason would be that the signal distribution in the present model allows for a drastic innovation.) For the special case with $n = 2$, part (b) of Proposition 4 above tells us that this cannot happen. Proposition 5 below lets us know whether Hansen's result can be overturned for larger markets.

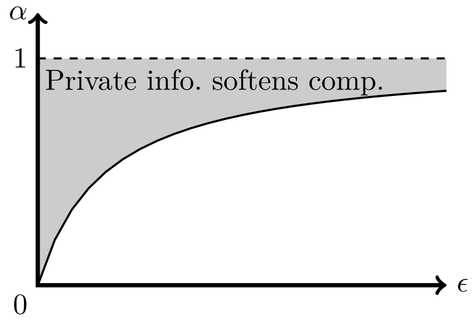

Figure 1: An illustration, in qualitative terms, of the results. Private information softens competition if costs are sufficiently common (high α) and/or if demand is sufficiently inelastic (low ε).

Proposition 5. ***(Private costs)*** *Consider the parameterized version of the model. Suppose $\alpha = 0$ and $\epsilon > 0$. Then, with incomplete information instead of complete information, the expected price is strictly lower ($\mathbb{E}\left[p^I\right] < \mathbb{E}\left[p^C\right]$).*

The relationship reported in Proposition 5 is in line with the one found in Hansen (1988). Thus, at least in the parameterized version of model, Hansen's assumption about the cost distribution—implying that the lowest-cost firm's optimal monopoly price always exceeds the second-lowest cost draw—does not matter for the results. The result in Proposition 5 also holds for any arbitrary number of firms, whereas Hansen assumed a duopoly market. We can conclude that, both in Hansen's (1988) model and in the present environment with private costs, asymmetric information is not anti-competitive but pro-competitive, at least in the sense that the firms' equilibrium mark-ups are (in expectation) smaller in an environment with asymmetric information.

4 Conclusions

This paper asked the question whether cost uncertainty softens competition in a homogeneous-good Bertrand model. Previous literature has studied this question in a framework with independent cost draws and found that the answer is "no." In contrast, the present paper showed that when the cost draws are interdependent to a sufficiently high degree, then cost uncertainty indeed softens competition. Specifically, under those conditions the expected price is higher and the expected consumer surplus is smaller with incomplete information than with complete information.

The paper also presented a simple and tractable—yet quite rich—parameterized model of price competition with privately known but interdependent cost draws. This model may be useful in applications. For example, it would be interesting, using an infinite-horizon repeated-game version of this framework, to study the effect of cost interdependence on the firms' ability to collude.

Appendix

Preliminaries

The density functions of the first-order and second-order statistics are, respectively, given by

$$f_{s_{(1)}} = nf\left(s_{(1)}\right)\left[1 - F\left(s_{(1)}\right)\right]^{n-1} \qquad \text{and} \qquad f_{s_{(2)}} = n(n-1)f\left(s_{(2)}\right)F\left(s_{(2)}\right)\left[1 - F\left(s_{(2)}\right)\right]^{n-2}. \tag{15}$$

Similarly, under the assumption that $n = 2$, the joint density of the first-order and second-order statistics is

$$2f\left(s_{(1)}\right)f\left(s_{(2)}\right) \qquad \text{if } s_{(1)} \leq s_{(2)} \text{ and } 0 \text{ otherwise.} \tag{16}$$

The above results can be found in, for example, Gumbel (1958/2004, p. 53), Gut (2009, pp. 102-105), or Wolfstetter (1999, p. 344).

Under the assumption of the parameterized model that $F(s)$ is given by (10), the expected values of a signal s, the first-order statistic $s_{(1)}$, and the second-order statistic $s_{(2)}$ are, respectively, given by

$$\begin{gathered}\mathbb{E}\left[s\right] = \int_0^1 sf(s)ds = \frac{1}{1+x}, \qquad \mathbb{E}\left[s_{(1)}\right] = \int_0^1 s_{(1)}f_{s_{(1)}}ds_{(1)} = \frac{1}{1+nx}, \\ \mathbb{E}\left[s_{(2)}\right] = \int_0^1 s_{(2)}f_{s_{(2)}}ds_{(2)} = \frac{1+(2n-1)x}{(1+nx)\left[1+(n-1)x\right]}.\end{gathered} \tag{17}$$

Finally in this subsection with preliminaries, consider the following lemma (to be used in the proof of Proposition 2).

Lemma A1. *We have*

$$\int_0^1 (n-1)f(s)\left[1-F(s)\right]^{n-2}F(s)\frac{\int_s^1 zf(z)\,dz}{1-F(s)}ds = \frac{(n-1)\,\mathbb{E}\left[s\right] - \mathbb{E}\left[s_{(1,n-1)}\right]}{(n-1)(n-2)} - \frac{\mathbb{E}\left[s_{(2)}\right]}{n(n-1)}. \tag{18}$$

Proof. First note that, by integrating by parts, we can write

$$\begin{aligned}&\int_0^1 (n-1)f(s)\left[1-F(s)\right]^{n-2}\frac{\int_s^1 zf(z)\,dz}{1-F(s)}ds \\ =\ & -\left[1-F(s)\right]^{n-1}\frac{\int_s^1 zf(z)\,dz}{1-F(s)}\Big|_0^1 + \int_0^1 \left[1-F(s)\right]^{n-1}\frac{-sf(s)\left[1-F(s)\right]+f(s)\int_s^1 zf(z)\,dz}{\left[1-F(s)\right]^2}ds \\ =\ & \mathbb{E}\left[s\right] - \int_0^1 sf(s)\left[1-F(s)\right]^{n-2}ds + \int_0^1 f(s)\left[1-F(s)\right]^{n-2}\frac{\int_s^1 zf(z)\,dz}{1-F(s)}ds.\end{aligned}$$

From (15) we have that the density function of $s_{(1)}$ is given by $nf(s)\left[1-F(s)\right]^{n-1}$. Therefore,

$$\int_0^1 sf(s)\left[1-F(s)\right]^{n-2}ds = (n-1)^{-1}\mathbb{E}\left[s_{(1,n-1)}\right],$$

where $\mathbb{E}\left[s_{(1,n-1)}\right]$ is the expected value of the first-order statistic among $n-1$ draws. We can thus equivalently write the above equality as

$$\int_0^1 f(s)\left[1-F(s)\right]^{n-2}\frac{\int_s^1 zf(z)\,dz}{1-F(s)}ds = \frac{\mathbb{E}\left[s\right] - (n-1)^{-1}\mathbb{E}\left[s_{(1,n-1)}\right]}{n-2}. \tag{19}$$

Next, integrating by parts yields

$$
\begin{aligned}
& \int_0^1 (n-1) f(s) [1-F(s)]^{n-2} F(s) \frac{\int_s^1 z f(z)\, dz}{1-F(s)} ds \\
= & -[1-F(s)]^{n-1} F(s) \frac{\int_s^1 z f(z)\, dz}{1-F(s)} \Big|_0^1 + \int_0^1 [1-F(s)]^{n-1} h(s) \left[\frac{\int_s^1 z f(z)\, dz}{1-F(s)} - sF(s) \right] ds \\
= & \int_0^1 f(s) [1-F(s)]^{n-2} \frac{\int_s^1 z f(z)\, dz}{1-F(s)} ds - \int_0^1 s f(s) F(s) [1-F(s)]^{n-2} ds \\
= & \frac{(n-1)\mathbb{E}[s] - \mathbb{E}\left[s_{(1,n-1)}\right]}{(n-1)(n-2)} - \frac{E\left[s_{(2)}\right]}{n(n-1)},
\end{aligned}
$$

where the last equality uses (19) and the density function of $s_{(2)}$, stated in (15). $\square$

Derivation of a Firm's Expected Cost

Using (2), firm i's cost can be written as $c_i = \left(1 - \alpha + \frac{\alpha}{n}\right) s_i + \frac{\alpha}{n} \sum_{j \neq i} s_j$. From the perspective of a firm that knows that it has the lowest signal, each one of the other firms' signals is distributed according to the density $f(s) / [1 - F[\chi(p_i)]]$, with support $[\chi(p_i), 1]$. Moreover, the other firms' signals are by assumption independent of each other. Thus, a firm's expected cost, conditional on the own signal s_i and on having chosen the lowest price, equals

$$
\widehat{\Phi}[\chi(p_i), s_i] \stackrel{\text{def}}{=} \mathbb{E}[c_i \mid s_i, p_i < p_j \text{ all } j \neq i] = \left(1 - \frac{\alpha(n-1)}{n}\right) s_i + \frac{\alpha(n-1)}{n} \left[\frac{\int_{\chi(p_i)}^1 z f(z) dz}{1 - F[\chi(p_i)]} \right], \tag{20}
$$

which simplifies to the expression in (4). Let $\Phi(s)$ denote the function we obtain when evaluating (20) at symmetry:

$$
\Phi(s) \stackrel{\text{def}}{=} \widehat{\Phi}(s, s) = s + \frac{\alpha(n-1)}{n} \left[\frac{\int_s^1 z f(z) dz}{1 - F(s)} - s \right]. \tag{21}
$$

Now, differentiate the function in (20) w.r.t. χ:

$$
\widehat{\phi}[\chi(p_i)] \stackrel{\text{def}}{=} \frac{\partial \widehat{\Phi}[\chi(p_i), s_i]}{\partial \chi} = \frac{\alpha(n-1) h[\chi(p_i)]}{n} \left[\frac{\int_{\chi(p_i)}^1 z f(z) dz}{1 - F[\chi(p_i)]} - \chi(p_i) \right]. \tag{22}
$$

Let $\phi(s)$ denote the function we obtain when evaluating (22) at symmetry:

$$
\phi(s) \stackrel{\text{def}}{=} \widehat{\phi}(s) = \frac{\alpha(n-1) h(s)}{n} \left[\frac{\int_s^1 z f(z) dz}{1 - F(s)} - s \right] = h(s) [\Phi(s) - s], \tag{23}
$$

where the last equality follows from (21).

Proof of Proposition 1

The proof begins by establishing that the boundary condition that is stated in the proposition must hold at any equilibrium. First note that, at an equilibrium, $p^*(1) \leq 1$ must hold. For if $p^*(1) > 1$, a standard Bertrand argument would apply and a firm with $s_i = 1$ could profitably deviate downwards. This is because (i) at the supposed equilibrium, the firm wins with zero probability and thus earns zero profits; and (ii) by charging a slightly lower price, it wins with positive probability while still having a positive price-cost margin (recall that $\Phi(1) = 1$), hence earning a positive profit.

Next note that, at an equilibrium, $p^*(1) \geq 1$ must hold. For if $p^*(1) < 1$, then (by $\Phi(1) = 1$ and by continuity of p^* and Φ) we must have $p^*(s) < \Phi(s)$ for any $s \in [\widehat{s}, 1]$, with $\widehat{s} < 1$. Moreover, for any $s \in [\widehat{s}, 1)$, the firm would win the market with a positive probability and thus also make a loss. This could be avoided by raising the price. The two observations ($p^*(1) \leq 1$ and $p^*(1) \geq 1$) imply that the boundary condition must hold.

The remainder of the proof draws on arguments in the proof of Theorem 2 in Maskin and Riley (1984). By the revelation principle, the equilibrium strategy can be represented as a direct mechanism that is incentive compatible and individually rational. Let $\pi(z,s)$ represent the expected profit of a type s firm that acts as a type z firm,

$$\pi(z,s)=\left[p(z)-\widehat{\Phi}\left(z,s\right)\right]D\left[p(z)\right]G\left(z\right). \tag{24}$$

Incentive compatibility requires that $\pi(s)\stackrel{\text{def}}{=}\pi(s,s)\geq\pi(z,s)$ for all $z\in[0,1]$. Individual rationality requires that $\pi(s)\geq 0$ for all $s\in[0,1]$.

Given the boundary condition $p(1)=1$, we have $\pi(1)=0$. Moreover, by the envelope theorem, $\pi_2(z,s)=-\widehat{\Phi}_2\left(z,s\right)D\left[p(z)\right]G\left(z\right)<0$, where we used the fact that $\widehat{\Phi}_2\left(z,s\right)>0$ (cf. (20)). It follows that $\pi(s)\geq 0$ for all $s\in[0,1]$, thus ensuring that individual rationality holds; in addition, we have $\pi(s)>0$ and $p(s)>\Phi\left(s\right)$ for all $s\in[0,1)$, which were two of the claims in the proposition.

To verify incentive compatibility, assume that $p(z)$ is differentiable and differentiate:

$$\begin{aligned}\frac{\partial\pi(z,s)}{\partial z}&=\left[\left(p'(z)-\frac{\partial\widehat{\Phi}\left(z,s\right)}{\partial z}\right)D\left[p(z)\right]+\left(p(z)-\widehat{\Phi}\left(z,s\right)\right)D'(p)p'(z)\right]G(z)\\&+\left[p(z)-\widehat{\Phi}\left(z,s\right)\right]D\left[p(z)\right]g\left(z\right)\\&=\left[D(p)+\left(p(z)-\widehat{\Phi}\left(z,s\right)\right)D'(p)\right]G(z)\times\left[p'(z)+T(z,s\right],\end{aligned} \tag{25}$$

where

$$T(z,s)\stackrel{\text{def}}{=}\frac{\left[\left(p(z)-\widehat{\Phi}\left(z,s\right)\right)\frac{g(z)}{G(z)}-\frac{\partial\widehat{\Phi}(z,s)}{\partial z}\right]D(p)}{D(p)+\left[p(z)-\widehat{\Phi}\left(z,s\right)\right]D'(p)}$$

and $g\left(z\right)=G'\left(z\right)$. Note that $T(z,s)$ is strictly increasing in s:

$$\frac{\partial T(z,s)}{\partial s}=-\frac{\frac{\partial\widehat{\Phi}(z,s)}{\partial s}\left[D(p)\frac{g(z)}{G(z)}+D'(p)\frac{\partial\widehat{\Phi}(z,s)}{\partial z}\right]D(p)}{\left[D(p)+\left[p(z)-\widehat{\Phi}\left(z,s\right)\right]D'(p)\right]^2}>0, \tag{26}$$

as $\widehat{\Phi}\left(z,s\right)$ is strictly increasing in both its arguments (cf. (20) and (22)), $g(z)<0$, and $D'(p)\leq 0$. Let $p(s)$ be defined by the differential equation with boundary condition

$$p'(s)=-T(s,s)=-\frac{\left[\left(p(s)-\widehat{\Phi}\left(s,s\right)\right)\frac{g(s)}{G(s)}-\frac{\partial\widehat{\Phi}(s,s)}{\partial z}\right]D[p(s)]}{D[p(s)]+\left[p(s)-\widehat{\Phi}\left(s,s\right)\right]D'[p(s)]},\qquad p(1)=1,$$

which simplifies to the differential equation specified in (6). It follows from (25) and (26) that

$$\frac{\partial\pi(z,s)}{\partial z}\gtreqless 0\text{ as }z\lesseqgtr s.$$

Therefore, $z=s$ yields the global maximum of $\pi(z,s)$. □

Proof of Proposition 2

Under the assumptions of complete information and inelastic demand, the equilibrium price paid by the consumers equals the cost of the second-most efficient firm; that is, $p^C=(1-\alpha)\,s_{(2)}+\frac{\alpha}{n}\sum_{i=1}^n s_{(i)}$. Taking expectations yields

$$\mathbb{E}\left[p^C\right]=(1-\alpha)\,\mathbb{E}\left[s_{(2)}\right]+\alpha\mathbb{E}\left[s\right]. \tag{27}$$

For the case with incomplete information and inelastic demand, we can use results from the proof of Proposition 1 but with $D(p)\equiv 1$. Setting the derivative in (25) equal to zero, using $D(p)=1$ and $D'(p)=0$, we have

$$
\begin{aligned}
\frac{\partial \pi(z,s)}{\partial z} &= \left(p'(z) - \frac{\partial \widehat{\Phi}(z,s)}{\partial z}\right) G(z) + \left[p(z) - \widehat{\Phi}(z,s)\right] g(z) = 0 \\
\Leftrightarrow \frac{\partial \left[p(z)\, G(z)\right]}{\partial z} &= \frac{\partial \widehat{\Phi}(z,s)}{\partial z} G(z) + \widehat{\Phi}(z,s)\, g(z).
\end{aligned} \tag{28}
$$

Evaluating the above first-order condition at the symmetric optimum ($z = s$), we can write

$$
\begin{aligned}
\frac{\partial \left[p(s)\, G(s)\right]}{\partial s} &= \phi(s)\, G(s) + \Phi(s)\, g(s) \\
&= h(s)\left[\Phi(s) - s\right] G(s) - (n-1)\Phi(s) G(s)\, h(s) \\
&= -h(s) G(s)\left[s + (n-2)\Phi(s)\right],
\end{aligned}
$$

where the second equality uses (23) and the relationship $g(s) = -(n-1)h(s)G(s)$. Integrating, we thus obtain

$$
p^I(s) = \frac{1}{G(s)} \int_s^1 h(z) G(z)\left[z + (n-2)\Phi(z)\right] dz, \tag{29}
$$

where the integration constant has been set equal to zero (this is required by the boundary condition $p(1) = 1$).

We can now compute the expected value of p^I. Using the density of the first-order statistic stated in (15), we can write

$$
\begin{aligned}
\mathbb{E}\left[p^I\right] &= \int_0^1 nf(s)\left[1 - F(s)\right]^{n-1} p^I(s) ds = n\int_0^1 f(s) \int_s^1 h(z) G(z)\left[z + (n-2)\Phi(z)\right] dz ds \\
&= n\int_0^1 F(s)\, h(s) G(s)\left[s + (n-2)\Phi(s)\right] ds = n\int_0^1 f(s)\, F(s)\left[1 - F(s)\right]^{n-2}\left[s + (n-2)\Phi(s)\right] ds \\
&= n\int_0^1 f(s)\, F(s)\left[1 - F(s)\right]^{n-2}\left[(n-1)s + \frac{\alpha(n-1)(n-2)}{n}\left[\frac{\int_s^1 z f(z) dz}{1 - F(s)} - s\right]\right] ds \\
&= \left(1 - \frac{\alpha(n-2)}{n}\right)\mathbb{E}\left[s_{(2)}\right] + \alpha(n-1)(n-2)\int_0^1 f(s)\, F(s)\left[1 - F(s)\right]^{n-2}\left[\frac{\int_s^1 z f(z) dz}{1 - F(s)}\right] ds \\
&= \left(1 - \frac{\alpha(n-2)}{n-1}\right)\mathbb{E}\left[s_{(2)}\right] + \alpha\, \mathbb{E}[s] - \frac{\alpha}{n-1}\mathbb{E}\left[s_{(1,n-1)}\right] \\
&= (1 - \alpha)\, \mathbb{E}\left[s_{(2)}\right] + \alpha\, \mathbb{E}[s] + \frac{\alpha}{n-1}\left[\mathbb{E}\left[s_{(2)}\right] - \mathbb{E}\left[s_{(1,n-1)}\right]\right],
\end{aligned}
$$

where the second equality uses (29), the third equality uses integration by parts, the fourth equality uses the definitions of $h(s)$ and $G(s)$, the fifth equality uses the expression for $\Phi(s)$ from (21), the sixth equality uses the density of the second-order statistic stated in (15), and the seventh equality uses Lemma A1. The last line together with (27) yield the first equality in (9). The second equality in (9) follows from noting that, by integrating by parts, we can write

$$
\mathbb{E}\left[s_{(2)}\right] = \int_0^1 \left[1 - F(s)\right]^n ds + n\int_0^1 F(s)\left[1 - F(s)\right]^{n-1} ds, \qquad \mathbb{E}\left[s_{(1,n-1)}\right] = \int_0^1 \left[1 - F(s)\right]^{n-1} ds.
$$

The claim in part (i) of the proposition, about the expected prices, follows immediately from (9). Next consider the claim in (ii) that $\mathbb{E}\left[S^I\right] \leq \mathbb{E}\left[S^C\right]$, with a strict inequality if and only if $\alpha > 0$. But this claim follows immediately from the expected price comparison in part (i) and the fact that, with inelastic demand, consumer surplus is given by $S = 1 - p$ (so it is linear and decreasing in the price). Similarly, realized industry profits given inelastic demand equal $p - c_{(1)}$ (so they are linear and increasing in the price). This means that also claim (iii) about profits follows immediately from the expected price comparison in part (i). Finally, given the expression for consumer surplus and realized profits above, it is clear that welfare (the sum of those terms) is independent of the price. Hence the welfare result, claim (iv), must also hold. □

Proof of Claims in Footnote 12

The equilibrium price in this version of the model with naive firms, $p_N^I(s)$, is obtained by first setting $\frac{\partial \widehat{\Phi}_N(s)}{\partial z} = 0$ in (28), and then rewriting the resulting expression similarly to how (28) was rewritten. Doing this yields

$$p_N^I(s) = \frac{1}{G(s)} \int_s^1 h(z) G(z)(n-1)\Phi_N(z) dz. \tag{30}$$

We can compute the expected value of p_N^I by using the density of the first-order statistic stated in (15):

$$\begin{aligned}
\mathbb{E}\left[p_N^I\right] &= \int_0^1 nf(s)[1-F(s)]^{n-1} p_N^I(s) ds = n\int_0^1 f(s)\int_s^1 h(z)G(z)(n-1)\Phi_N(z) dz ds \\
&= n\int_0^1 F(s) h(s) G(s)(n-1)\Phi_N(s) ds = n\int_0^1 f(s) F(s)[1-F(s)]^{n-2}(n-1)\Phi_N(s) ds \\
&= n(n-1)\int_0^1 f(s) F(s)[1-F(s)]^{n-2}\left[s + \frac{\alpha(n-1)}{n}\left[\mathbb{E}[s]-s\right]\right] ds \\
&= \left(1-\frac{\alpha(n-1)}{n}\right)\mathbb{E}\left[s_{(2)}\right] + \alpha(n-1)^2\mathbb{E}[s]\int_0^1 f(s) F(s)[1-F(s)]^{n-2} ds \\
&= \left(1-\frac{\alpha(n-1)}{n}\right)\mathbb{E}\left[s_{(2)}\right] + \frac{\alpha(n-1)}{n}\mathbb{E}[s] \\
&= (1-\alpha)\mathbb{E}\left[s_{(2)}\right] + \alpha\mathbb{E}[s] + \frac{\alpha}{n}\left[\mathbb{E}\left[s_{(2)}\right]-\mathbb{E}[s]\right],
\end{aligned}$$

where the second equality uses (30), the third equality uses integration by parts, the fourth equality uses the definitions of $h(s)$ and $G(s)$, the fifth equality uses the expression for $\Phi_N(s)$ from footnote 12, and the sixth and seventh equalities use the density of the second-order statistic stated in (15).

Proof of Proposition 3

From Proposition 1 we know that, for all $s \in (0,1)$ and all $\alpha \in [0,1]$, $p^I(s) > \Phi(s)$. Thus, taking expectations over $s_{(1)}$, we have $\mathbb{E}\left[p^I(s_{(1)})\right] > \mathbb{E}\left[\Phi\left(s_{(1)}\right)\right]$ for all $\alpha \in [0,1]$. Computing $\mathbb{E}\left[\Phi\left(s_{(1)}\right)\right]$, we obtain

$$\begin{aligned}
\mathbb{E}\left[\Phi\left(s_{(1)}\right)\right] &= \int_0^1 nf(s)[1-F(s)]^{n-1}\Phi(s) ds = \int_0^1 nf(s)[1-F(s)]^{n-1}\left[s + \frac{\alpha(n-1)}{n}\left(\frac{\int_s^1 zf(z)dz}{1-F(s)} - s\right)\right] ds \\
&= \left(1-\frac{\alpha(n-1)}{n}\right)\mathbb{E}\left[s_{(1)}\right] + \alpha(n-1)\int_0^1 f(s)[1-F(s)]^{n-2}\int_s^1 zf(z) dz ds \\
&= \left(1-\frac{\alpha(n-1)}{n}\right)\mathbb{E}\left[s_{(1)}\right] + \alpha\left[-[1-F(s)]^{n-1}\int_s^1 zf(z) dz \,|_0^1 - \int_0^1 sf(s)[1-F(s)]^{n-1} ds\right] \\
&= \left(1-\frac{\alpha(n-1)}{n}\right)\mathbb{E}\left[s_{(1)}\right] + \alpha\left[\mathbb{E}[s] - \frac{1}{n}\mathbb{E}\left[s_{(1)}\right]\right] \\
&= (1-\alpha)\mathbb{E}\left[s_{(1)}\right] + \alpha\mathbb{E}[s],
\end{aligned} \tag{31}$$

where the first equality uses the density function of the first-order statistic stated in (15), the second equality uses the expression for $\Phi(s)$ in (21), the third equality rewrites and again uses (15), the fourth equality uses integration by parts, and the fifth equality yet again uses (15).

For $\alpha = 1$, all firms have the same cost; thus, for $\alpha = 1$ we have $\mathbb{E}\left[p^C(s)\right] = \mathbb{E}[s] = \mathbb{E}[\Phi(s)]$, where the last equality follows from (31). This means that $\mathbb{E}\left[p^I(s)\right] > \mathbb{E}\left[p^C(s)\right]$ for $\alpha = 1$. Moreover, both $\mathbb{E}\left[p^I(s)\right]$ and $\mathbb{E}\left[p^C(s)\right]$ are continuous in α, which means that there exists some $\alpha' < 1$ such that $\mathbb{E}\left[p^I(s)\right] > \mathbb{E}\left[p^C(s)\right]$ for all $\alpha \in [\alpha', 1]$. This establishes the claim made in part (i).

Next turn to the claim made in part (ii). Let $\mathbb{E}_{s_{-i}}$ denote the operator that takes expectations over all $s_j \neq s_i$. For $\alpha = 1$, we can write

$$\mathbb{E}_{s_{-i}}\left[S\left[p^C(s)\right]\right] > S\left[\mathbb{E}_{s_{-i}}\left[p^C(s)\right]\right] = S\left[\mathbb{E}_{s_{-i}}\left[\Phi(s)\right]\right] \geq S\left[p^I(s_i)\right] \quad \text{for all } s_i,$$

where the first inequality follows from strict convexity of S and Jensen's inequality, the equality follows from the results in the proof of part (i), and the last inequality follows from $S' < 0$, the expected profit expression in (3), and the fact that the expected equilibrium profits (conditional on s_i) must be non-negative. By taking expectations also over s_i, we obtain $\mathbb{E}\left[S\left[p^C(s)\right]\right] > \mathbb{E}\left[S\left[p^I(s)\right]\right]$ for $\alpha = 1$. Thus, by continuity of the relevant functions, there exists some $\alpha'' < 1$ such that $\mathbb{E}\left[S\left[p^C(s)\right]\right] > \mathbb{E}\left[S\left[p^I(s)\right]\right]$ for all $\alpha \in [\alpha'', 1]$.

Finally turn to the claim in part (iii). For $\alpha = 1$, all firms have the same cost; thus, under complete information, Bertrand competition yields a realized profit of zero for all signals. In contrast, under incomplete information the winning firm's profit, conditional on the own signal, is strictly positive for all $s_i < 1$ (see Proposition 1). Taking expectations, we thus have that $\mathbb{E}\left[\Pi^I\right] > \mathbb{E}\left[\Pi^C\right]$ for $\alpha = 1$. Thus, by continuity of the relevant functions, there exists some $\alpha''' < 1$ such that $\mathbb{E}\left[\Pi^I\right] > \mathbb{E}\left[\Pi^C\right]$ for $\alpha = 1$ for all $\alpha \in [\alpha''', 1]$.

The notation $\widehat{\alpha}$ used in the proposition is defined as $\widehat{\alpha} = \max\{\alpha', \alpha'', \alpha'''\}$. □

Proof of Corollary 1

By making use of the assumed functional forms, we can write

$$\int_s^1 zf(z)dz = s(1-s)^x + \frac{(1-s)^{1+x}}{1+x}$$

and thus

$$\Phi(s) = s + \frac{\alpha(n-1)}{n}\left[\frac{\int_s^1 zf(z)dz}{1-F(s)} - s\right] = s + \frac{\alpha(n-1)}{n}\frac{1-s}{1+x}. \tag{32}$$

Using the assumption $p(s) = A + (1-A)s$, this in turn means that we have

$$p(s) - \Phi(s) = \left[A - \frac{\alpha(n-1)}{n(1+x)}\right](1-s), \tag{33}$$

$$(n-1)p(s) - s - (n-2)\Phi(s) = (n-1)\left[A - \frac{\alpha(n-2)}{n(1+x)}\right](1-s). \tag{34}$$

Evaluating the specified demand function and its derivative at $p(s) = A + (1-A)s$, we obtain

$$D[p(s)] = (1-A)^{\varepsilon}(1-s)^{\varepsilon} \qquad \text{and} \qquad D'[p(s)] = -\varepsilon(1-A)^{\varepsilon-1}(1-s)^{\varepsilon-1}. \tag{35}$$

Finally note that with the specified functional forms, we have $h(s) = x/(1-s)$ and $p'(s) = 1-A$.

Now plug the expressions above into the differential equation in (6):

$$1 - A = \frac{(n-1)\left[A - \frac{\alpha(n-2)}{n(1+x)}\right](1-s)\frac{x}{1-s}(1-A)^{\varepsilon}(1-s)^{\varepsilon}}{(1-A)^{\varepsilon}(1-s)^{\varepsilon} - \left[A - \frac{\alpha(n-1)}{n(1+x)}\right](1-s)\varepsilon(1-A)^{\varepsilon-1}(1-s)^{\varepsilon-1}}.$$

The root $A = 1$ would not yield a strictly increasing price; thus, we can suppose that $A \neq 1$ and the above equality simplifies to

$$1 = \frac{(n-1)\left[A - \frac{\alpha(n-2)}{n(1+x)}\right]x}{1 - A - \left[A - \frac{\alpha(n-1)}{n(1+x)}\right]\varepsilon} \Leftrightarrow A = \frac{n(1+x) + \alpha(n-1)\left[\epsilon + (n-2)x\right]}{n(1+x)\left[1 + \epsilon + (n-1)x\right]}.$$

The last term can be rewritten as the expression for A stated in (11). □

Proof of Lemma 1

We can write

$$\begin{aligned} c_{(2)} \leq p^m\left[c_{(1)}\right] &\Leftrightarrow \varepsilon\left(c_{(2)}-c_{(1)}\right) \leq 1-c_{(2)} \\ &\Leftrightarrow \varepsilon(1-\alpha)\left[s_{(2)}-s_{(1)}\right] \leq 1-(1-\alpha)s_{(2)}-\frac{\alpha}{n}\sum_{j=1}^{n} s_{(j)} \\ &\Leftrightarrow \left[\varepsilon(1-\alpha)-\frac{\alpha}{n}\right]\left[s_{(2)}-s_{(1)}\right] \leq 1-\left[1-\frac{(n-1)\alpha}{n}\right]s_{(2)}-\frac{\alpha}{n}\sum_{j=2}^{n} s_{(j)}, \end{aligned} \tag{36}$$

where the first line uses (13) and second line uses the definitions of $c_{(1)}$ and $c_{(2)}$ in (2). If (14) holds, then the left-hand side of the last inequality in (36) is weakly negative. The right-hand side, however, is weakly positive for all realizations of $s_{(j)}$. This proves the "if" part of the claim. The "only if" part follows from noting that if (14) is violated, then the last inequality in (36) fails to hold if $s_{(1)}$ is close enough to zero and if $s_{(j)}$ (for all $j \geq 2$) are close enough to one. □

Proof of Proposition 4

Consider claim (a) in the proposition. From Lemma 1 we know that, for all $\alpha \geq \frac{n\varepsilon}{1+n\varepsilon}$, we have $p^C = c_{(2)}$ and that we therefore can write $\mathbb{E}[p^C] = \mathbb{E}[c_{(2)}] = (1-\alpha)\mathbb{E}[s_{(2)}] + \alpha\mathbb{E}[s]$. Using this expression and (11), we can also, for all $\alpha \geq \frac{n\varepsilon}{1+n\varepsilon}$, write

$$\mathbb{E}[p^I] > \mathbb{E}[p^C] \Leftrightarrow A + (1-A)\mathbb{E}[s_{(1)}] > (1-\alpha)\mathbb{E}[s_{(2)}] + \alpha\mathbb{E}[s] \Leftrightarrow A > \frac{(1-\alpha)\mathbb{E}[s_{(2)}] + \alpha\mathbb{E}[s] - \mathbb{E}[s_{(1)}]}{1-\mathbb{E}[s_{(1)}]}.$$

By using (11) and (17), the last inequality above can be rewritten as

$$\begin{aligned} &\frac{1}{1+\epsilon+(n-1)\,x} + \frac{\alpha\,(n-1)\,[\epsilon+(n-2)x]}{n\,(1+x)\,[1+\epsilon+(n-1)\,x]} > \frac{1}{1+(n-1)\,x} + \frac{\alpha\left[(n-1)^2\,x-(1+nx)\right]}{n\,(1+x)\,[1+(n-1)\,x]} \Leftrightarrow \\ &\frac{\alpha}{n\,(1+x)}\left[\frac{(n-1)\,[\epsilon+(n-2)x]}{1+\epsilon+(n-1)\,x} - \frac{(n-1)^2\,x-(1+nx)}{1+(n-1)\,x}\right] > \frac{1}{1+(n-1)\,x} - \frac{1}{1+\epsilon+(n-1)\,x}. \end{aligned} \tag{37}$$

One can show that the expression in large square brackets on the left-hand side of (37) is strictly positive (it is increasing in ϵ and strictly positive for $\epsilon = 0$). Hence, (37) holds for all $\alpha \geq n\epsilon/(1+n\epsilon)$ if it holds when evaluated at $\alpha = n\epsilon/(1+n\epsilon)$. Plugging $\alpha = n\epsilon/(1+n\epsilon)$ into (37) and rewriting the right-hand side, we have

$$\begin{aligned} &\frac{\epsilon}{(1+n\epsilon)\,(1+x)}\left[\frac{(n-1)\,[\epsilon+(n-2)x]}{1+\epsilon+(n-1)\,x} - \frac{(n-1)^2\,x-(1+nx)}{1+(n-1)\,x}\right] > \frac{\epsilon}{[1+(n-1)\,x]\,[1+\epsilon+(n-1)\,x]} \Leftrightarrow \\ &(n-1)\,[\epsilon+(n-2)x]\,[1+(n-1)\,x] - \left[(n-1)^2\,x-(1+nx)\right][1+\epsilon+(n-1)\,x] > (1+n\epsilon)\,(1+x)\,. \end{aligned}$$

The left-hand side of the last inequality can be written as $(1+x)\,[1+(n-1)x+n\epsilon]$, which means that the term $(1+x)$ cancels out and it is then easy to see that the inequality always holds. This establishes part (a) of the proposition.

Now turn to the claims in part (b) of the proposition. All those claims, except for the one about the limit value, follow if we can show that, given $n = 2$, $\mathbb{E}[p^I]$ and $\mathbb{E}[p^C]$ have the following four properties:

(i) At $\alpha = 0$, $\mathbb{E}[p^I] < \mathbb{E}[p^C]$.

(ii) For all $\alpha \in (0, 1)$, $\mathbb{E}[p^I]$ is linear and strictly increasing in α.

(iii) For all $\alpha \in \left[\frac{2\varepsilon}{1+2\varepsilon}, 1\right]$, $\mathbb{E}[p^I] > \mathbb{E}[p^C]$.

(iv) For all $\alpha \in \left(0, \frac{2\varepsilon}{1+2\varepsilon}\right)$, $\mathbb{E}[p^C]$ is strictly concave in α.

Property (i) is shown in the proof of Proposition 5 below. Similarly, property (iii) is implied by the (a) part of Proposition 4, which was proven above. To prove property (ii), first note that it follows immediately from

(11) that $\mathbb{E}[p^I]$ is a linear (affine) function of α. Moreover we have

$$\frac{\partial A}{\partial \alpha} = -\frac{n-1}{n[1+\varepsilon+(n-1)x]} + \frac{n-1}{n(1+x)},$$

which by inspection is strictly positive for all $\varepsilon > 0$ and all $n \geq 2$.

Finally, to prove property (iv) first note that, with $n = 2$, eq. (36) yields

$$c_{(2)} \leq p^m \left[c_{(1)}\right] \stackrel{\text{def}}{=} \frac{1+\varepsilon c_{(1)}}{1+\varepsilon} \Leftrightarrow s_{(2)} \leq \varphi\left[s_{(1)}\right] \stackrel{\text{def}}{=} \frac{1+\left[\epsilon\left(1-\alpha\right)-\frac{\alpha}{2}\right] s_{(1)}}{\left(1+\epsilon\right)\left(1-\alpha\right)+\frac{\alpha}{2}}.$$

The joint density function of $s_{(1)}$ and $s_{(2)}$ is stated in (16). Using this we can, for $n = 2$ and $\alpha < \frac{2\varepsilon}{1+2\varepsilon}$, write

$$\mathbb{E}\left[p^C\right] = 2\int_0^1 f\left[s_{(1)}\right]\left[\int_{s_{(1)}}^{\varphi\left[s_{(1)}\right]} c_{(2)} f\left[s_{(2)}\right] ds_{(2)} + \int_{\varphi\left[s_{(1)}\right]}^1 p^m\left[c_{(1)}\right] f\left[s_{(2)}\right] ds_{(2)}\right] ds_{(1)}.$$

Differentiating with respect to α yields

$$\frac{\partial \mathbb{E}\left[p^C\right]}{\partial \alpha} = 2\int_0^1 f\left[s_{(1)}\right]\left[\int_{s_{(1)}}^{\varphi\left[s_{(1)}\right]} \frac{\partial c_{(2)}}{\partial \alpha} f\left[s_{(2)}\right] ds_{(2)} + \int_{\varphi\left[s_{(1)}\right]}^1 \frac{\partial p^m\left[c_{(1)}\right]}{\partial \alpha} f\left[s_{(2)}\right] ds_{(2)}\right] ds_{(1)}, \tag{38}$$

where

$$\frac{\partial c_{(2)}}{\partial \alpha} = -\frac{s_{(2)}-s_{(1)}}{2} \quad \text{and} \quad \frac{\partial p^m\left[c_{(1)}\right]}{\partial \alpha} = \frac{\epsilon\left[s_{(2)}-s_{(1)}\right]}{2\left(1+\epsilon\right)}.$$

Note that, in (38), α appears only in the expression for $\varphi\left[s_{(1)}\right]$. Thus, differentiating a second time with respect to α yields

$$\frac{\partial^2 \mathbb{E}\left[p^C\right]}{\partial \alpha^2} = -\int_0^1 \left[\varphi\left(s_{(1)}\right) - s_{(1)} + \frac{\epsilon\left[\varphi\left(s_{(1)}\right)-s_{(1)}\right]}{1+\epsilon}\right] \frac{\partial \varphi\left[s_{(1)}\right]}{\partial \alpha} f\left[s_{(1)}\right] f\left[\varphi\left(s_{(1)}\right)\right] ds_{(1)} < 0,$$

where

$$\frac{\partial \varphi\left[s_{(1)}\right]}{\partial \alpha} = \frac{\left(\epsilon+\frac{1}{2}\right)\left[1-s_{(1)}\right]}{\left[\left(1+\epsilon\right)\left(1-\alpha\right)+\frac{\alpha}{2}\right]^2} > 0.$$

This yields the desired result.

It remains to show the claim that $\lim_{\epsilon\to 0} \alpha^* = 0$. However, this follows from the result that $\alpha^* < n\epsilon/(1+n\epsilon)$, in conjunction with fact that $\alpha^* \geq 0$.

To do the simulations presented in Fig. ??, we need an algebraic expression that implicitly characterizes the cutoff value α^*.

□

Proof of Proposition 5

For later reference, first note that, for an arbitrary n and with $\alpha = 0$, we can write

$$\mathbb{E}\left[p^C\right] = \int_0^1 \left[\int_{s_{(1)}}^{p^m\left[s_{(1)}\right]} s_{(2)} k\left[s_{(1)}, s_{(2)}\right] ds_{(2)} + \int_{p^m\left[s_{(1)}\right]}^1 p^m\left[s_{(1)}\right] k\left[s_{(1)}, s_{(2)}\right] ds_{(2)}\right] ds_{(1)}, \tag{39}$$

where

$$p^m\left[s_{(1)}\right] = \frac{1+\epsilon s_{(1)}}{1+\epsilon} \qquad \text{and} \qquad k\left[s_{(1)}, s_{(2)}\right] = n(n-1) f\left[s_{(1)}\right] f\left[s_{(2)}\right] \left[1-F\left(s_{(2)}\right)\right]^{n-2}$$

if $s_{(1)} \leq s_{(2)}$ and $k\left[s_{(1)}, s_{(2)}\right] = 0$ otherwise. Differentiating w.r.t. ϵ, we have

$$\frac{\partial \mathbb{E}\left[p^C\right]}{\partial \epsilon} = \int_0^1 \int_{p^m\left[s_{(1)}\right]}^1 \frac{\partial p^m\left[s_{(1)}\right]}{\partial \epsilon} k\left[s_{(1)}, s_{(2)}\right] ds_{(2)} ds_{(1)},$$

where

$$\frac{\partial p^m\left[s_{(1)}\right]}{\partial \epsilon}=-\frac{1-s_{(1)}}{(1+\epsilon)^2}.$$

Using the functional forms for the distributions, we get

$$\begin{aligned}
\frac{\partial \mathbb{E}\left[p^C\right]}{\partial \epsilon} &= -\int_0^1\left[n(n-1)x^2\int_{\frac{1+\epsilon s_{(1)}}{1+\epsilon}}^1\frac{1-s_{(1)}}{(1+\epsilon)^2}\left[1-s_{(1)}\right]^{x-1}\left[1-s_{(2)}\right]^{x-1}\left[1-s_{(2)}\right]^{x(n-2)}ds_{(2)}\right]ds_{(1)}\\
&= -\frac{n(n-1)x^2}{(1+\epsilon)^2}\int_0^1\left[1-s_{(1)}\right]^x\left[\int_{\frac{1+\epsilon s_{(1)}}{1+\epsilon}}^1\left[1-s_{(2)}\right]^{x(n-1)-1}ds_{(2)}\right]ds_{(1)}\\
&= \frac{nx}{(1+\epsilon)^2}\int_0^1\left[1-s_{(1)}\right]^x\left[\left[1-s_{(2)}\right]^{x(n-1)}|^1_{\frac{1+\epsilon s_{(1)}}{1+\epsilon}}\right]ds_{(1)}\\
&= -\frac{nx}{(1+\epsilon)^2}\int_0^1\left[1-s_{(1)}\right]^x\left[1-\frac{1+\epsilon s_{(1)}}{1+\epsilon}\right]^{x(n-1)}ds_{(1)}\\
&= -\frac{nx\epsilon^{x(n-1)}}{(1+\epsilon)^{2+x(n-1)}}\int_0^1\left[1-s_{(1)}\right]^{xn}ds_{(1)}\\
&= \frac{nx\epsilon^{x(n-1)}}{(nx+1)(1+\epsilon)^{2+x(n-1)}}\left[1-s_{(1)}\right]^{nx+1}|_0^1=-\frac{nx\epsilon^{x(n-1)}}{(nx+1)(1+\epsilon)^{2+x(n-1)}}. \qquad (40)
\end{aligned}$$

Differentiating a second time w.r.t. ϵ, we have

$$\frac{\partial^2 \mathbb{E}\left[p^C\right]}{\partial \epsilon^2}=-\frac{nx\epsilon^{x(n-1)-1}\left[x(n-1)-2\epsilon\right]}{(nx+1)(1+\epsilon)^{3+x(n-1)}}. \qquad (41)$$

Moreover, by (11) and (17) and with $\alpha=0$, the expected price under incomplete information equals

$$\begin{aligned}
\mathbb{E}\left[p^I\right] &= A+(1-A)\,\mathbb{E}\left[s_{(1)}\right]=\frac{1}{1+(n-1)\,x+\epsilon}+\left[1-\frac{1}{1+(n-1)\,x+\epsilon}\right]\frac{1}{1+nx}\\
&= \frac{1}{1+nx}+\frac{1}{1+(n-1)\,x+\epsilon}\left[1-\frac{1}{1+nx}\right]=\frac{1}{1+nx}+\frac{nx}{\left[1+(n-1)\,x+\epsilon\right](1+nx)}. \qquad (42)
\end{aligned}$$

We thus have

$$\frac{\partial \mathbb{E}\left[p^I\right]}{\partial \epsilon}=-\frac{nx}{\left[1+(n-1)\,x+\epsilon\right]^2(1+nx)},\qquad \frac{\partial^2 \mathbb{E}\left[p^I\right]}{\partial \epsilon^2}=\frac{2nx}{\left[1+(n-1)\,x+\epsilon\right]^3(1+nx)}. \qquad (43)$$

Now turn to the claim in the proposition. Let Δ denote the extent to which $\mathbb{E}\left[p^C\right]$ is larger than $\mathbb{E}\left[p^I\right]$:

$$\Delta(\epsilon,x,n)=\mathbb{E}\left[p^C\right]-\mathbb{E}\left[p^I\right].$$

The claim in the proposition will follow if we can show that $\Delta(\epsilon,x,n)$ has the following three properties:

(i) It equals zero at $\epsilon=0$: $\Delta(0,x,n)=0$.

(ii) In the limit as ϵ approaches infinity, it equals zero: $\lim_{\epsilon\to\infty}\Delta(\epsilon,x,n)=0$.

(iii) For all $\epsilon\in(1+(n-1)\,x,\infty)$, $\Delta(\epsilon,x,n)$ is decreasing in ϵ.

(iv) For all $\epsilon\in(0,1+(n-1)\,x)$, $\Delta(\epsilon,x,n)$ is quasiconcave in ϵ.

Property (i) follows immediately from (9). The fact that property (ii) holds follows from noting, from (11) and (39), that both $\mathbb{E}\left[p^I\right]$ and $\mathbb{E}\left[p^C\right]$ approach $\mathbb{E}[s_{(1)}]$ as $\epsilon\to\infty$. Next, consider property (iii). By using (40) and (43), we can write

$$\begin{aligned}
\frac{\partial\Delta(\epsilon,x,n)}{\partial\epsilon}\le 0 &\Leftrightarrow -\frac{nx\epsilon^{x(n-1)}}{(nx+1)(1+\epsilon)^{2+x(n-1)}}\le-\frac{nx}{\left[1+(n-1)\,x+\epsilon\right]^2(1+nx)}\\
&\Leftrightarrow \epsilon^{x(n-1)}\left[1+(n-1)\,x+\epsilon\right]^2\ge(1+\epsilon)^{2+x(n-1)} \qquad (44)\\
&\Leftrightarrow x(n-1)\ln(\epsilon)+2\ln\left[1+(n-1)\,x+\epsilon\right]-\left[2+x(n-1)\right]\ln(1+\epsilon)\overset{\text{def}}{=}\psi(\epsilon,x,n)\ge 0
\end{aligned}$$

By evaluating this expression at $\epsilon = 1 + (n-1)x$, we obtain

$$\psi[1 + (n-1)x, x, n] = \ln(4) + [2 + (n-1)x] \ln \left[\frac{1 + (n-1)\,x}{2 + (n-1)\,x} \right], \tag{45}$$

which can be shown to be positive (the expression in (45) equals zero when evaluated at $(n-1)x = 0$, it approaches the positive number $\ln(4) - 1$ as $(n-1)x$ approaches infinity, and it is strictly concave in $(n-1)x$). Moreover, $\psi(\epsilon, x, n)$ can be shown to approach zero as ϵ approaches infinity and, for all $\epsilon \geq 1 + (n-1)\,x$, to be decreasing in ϵ. It follows that $\Delta(\epsilon, x, n)$ is decreasing in ϵ for all $\epsilon > 1 + (n-1)\,x$.

Finally, consider property (iv). By using (41) and (43), we can write

$$\frac{\partial^2 \Delta(\epsilon, x, n)}{\partial \epsilon^2} = -\frac{nx\epsilon^{x(n-1)-1}\,[x(n-1) - 2\epsilon]}{(nx+1)\,(1+\epsilon)^{3+x(n-1)}} - \frac{2nx}{[1 + (n-1)\,x + \epsilon]^3\,(1+nx)}.$$

Hence,

$$\frac{\partial^2 \Delta(\epsilon, x, n)}{\partial \epsilon^2} \leq 0 \Leftrightarrow -\frac{\epsilon^{x(n-1)-1}\,[x(n-1) - 2\epsilon]}{(1+\epsilon)^{3+x(n-1)}} \leq \frac{2}{[1 + (n-1)\,x + \epsilon]^3}.$$

Evaluating at an ϵ that satisfies $\frac{\partial \Delta(\epsilon, x, n)}{\partial \epsilon} = 0$, thus using (44), we obtain

$$\begin{aligned} -\frac{\epsilon^{x(n-1)-1}\,[x(n-1) - 2\epsilon]}{(1+\epsilon)\,\epsilon^{x(n-1)}\,[1 + (n-1)\,x + \epsilon]^2} &\leq \frac{2}{[1 + (n-1)\,x + \epsilon]^3} \\ \Leftrightarrow -[x(n-1) - 2\epsilon]\,[1 + (n-1)\,x + \epsilon] &\leq 2\epsilon(1+\epsilon) \\ \Leftrightarrow 2\epsilon(n-1)x \leq x(n-1)\,[1 + (n-1)\,x + \epsilon] &\Leftrightarrow \epsilon \leq 1 + (n-1)\,x. \end{aligned}$$

That is, for all $\epsilon \in (0, 1 + (n-1)\,x)$, $\Delta(\epsilon, x, n)$ is quasiconcave in ϵ. □

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