Bertrand under Uncertainty: Private and Common Costs

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Abstract

This paper asks whether private information about costs in a homogeneous-good Bertrand model softens competition. Earlier literature has shown that the answer (perhaps counterintuitively) is "no," while assuming (i) private (i.e., independent) cost draws and (ii) no drastic innovations. I first show, in a fairly general setting, that by relaxing (i) and instead allowing for sufficiently much common (interdependent) cost draws, private information indeed softens competition. I then study a specification that yields a closed-form solution and show that relaxing (ii) but not (i) does not alter the result in the earlier literature. While relying on specific functional forms, this specification is quite rich and might be useful in applications. It allows for any (positive) degree of interdependence between the cost draws, for any demand elasticity, and for any number of firms. The closed-form solution is simple and in pure strategies.

Keywords: Bertrand competition, Hansen-Spulber model, private information, information sharing, common values, private values, winner's curse

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1 Introduction

The Bertrand model of price competition predicts that price equals marginal cost and that firms earn zero profit—a result which is often referred to as the Bertrand paradox, as it suggests that the presence of only two firms is sufficient to eliminate all market power and give rise to the perfectly competitive outcome. The paradox has prompted a number of scholars to study extensions and variations of Bertrand's original model. By doing that they have identified several model features that, if added to the standard setup, resolves the paradox by providing some amount of market power to the firms. Examples of such features include product differentiation, capacity constraints, repeated interaction, and cost asymmetries between firms.

In an interesting paper, Spulber (1995) studies another, empirically very plausible, variation of the standard Bertrand model, namely to assume that each price-setting firm has private information about some characteristic of its production technology, a leading example of which is the firm's (constant) marginal cost.[1] Spulber shows that in that setting there is a unique and symmetric equilibrium price, which is increasing in the own marginal cost. Importantly, the equilibrium price lies strictly between the marginal cost and the monopoly price, which means that the firms have some market power and earn a positive profit.[2]

What exactly is the model feature that gives rise to that outcome, thus solving the Bertrand paradox? An answer that naturally comes to mind is that it is asymmetric information (or uncertainty more generally), and this is exactly what Spulber (1995) suggests (p. 10, emphasis added):

> Asymmetric information thus plays an important role in imperfect competition. In [the model] studied here, the surgical precision that is required to price slightly below [...] higher cost rivals is eliminated by the lack of exact knowledge about the characteristics of the rivals. In the short run, with market structure fixed, *asymmetric information appears to reduce competition*[...].

It is not only Spulber himself who interprets his results in this fashion, so do several other authors. For example, Spiegel and Tookes (2008, p. 33, fn. 33) write that "Spulber (1995) also shows how, in Bertrand competition, not knowing rivals' costs implies equilibrium prices that are above marginal costs (i.e., information asymmetry softens product market competition)."[3]

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[1] The model that Spulber studies has also found its way into textbooks—see Wolfstetter (1999, pp. 236-37) and Belleflamme and Peitz (2015, pp. 47-49). Arozamena and Weinschelbaum (2009) study a sequential version of Spulber's model and compare with the simultaneous-move version. Lofaro (2002) obtains a closed-form solution of Spulber's model by assuming a uniform distribution, and he then compares the price competition outcomes with the quantity competition outcomes. Athey (2002, p. 198) generalizes some of Spulber's results in a number of directions, allowing for, among other things, asymmetric cost distributions. Abbink and Brandts (2007) test Spulber's model in the laboratory.

[2] The firm that draws the lowest marginal cost (and thus charges the lowest price) earns a positive profit ex post. The other firms earn a zero profit ex post, but their expected profit, at the stage before they have learned their cost parameter, is positive.

[3] See also Wolfstetter (1999), who presents two simple versions of the model, one with inelastic and one with elastic demand. He introduces the analysis of these models by stating that (p. 236): "A much simpler [relative to the model with capacity constraints] resolution of the Bertrand paradox can be found by introducing incomplete information." Yet another paper that refers (twice—on p. 638 and p. 646) to Spulber's (1995) result as a "resolution of the Bertrand paradox" is Abbink and Brandts (2007).

But is it really asymmetric information (or uncertainty) that softens competition? When Spulber assumes uncertainty about the firms' marginal cost parameters, he implicitly also introduces the assumption that these parameters may differ from each other. This means that, in principle, the model feature that creates market power could be cost heterogeneity and not uncertainty. Moreover, it follows from relatively early work of Hansen (1988) that, at least for a special case of Spulber's (1995) model, it is indeed cost heterogeneity that softens product market competition. Uncertainty is, in Hansen's (1988) setting and given the presence of cost heterogeneity, not anti-competitive but pro-competitive—at least in the sense that it lowers expected price and raises expected consumer and total surplus. Uncertainty also makes expected industry profits go up, which of course can be thought of as providing the firms with more market power.

However, Hansen's analysis is carried out under two assumptions that seem restrictive and which, if relaxed, conceivably could reverse his results. First, Hansen's setup does not allow for the possibility that a firm makes a "drastic innovation"—that is, that a firm's cost advantage is so large that, under complete information, it can optimally behave like a monopolist. Allowing for drastic innovations would tend to lower the expected price in the complete information model, which might overturn Hansen's results. Second, and most importantly, Hansen's (as well as Spulber's) analysis of the cost uncertainty model assumes that *the cost draws are independent*. Such an assumption indeed appears to be restrictive. Empirically we would expect firms' costs to be at least to some extent positively correlated (or interdependent), as all firms in a market are likely to be affected by external and economy-wide shocks to wages, interest rates, the price of material and energy, etc.

To understand why we should expect Hansen's results to be reversed if we assumed a sufficiently strong (positive) degree of interdependence between the costs, think of the extreme case where all firms in the market share the same (constant marginal) cost. First note that if this common cost is known, we are back to the Bertrand paradox and zero market power. Next suppose the cost is *not* known but each firm observes some private signal about its true value. If we could show that, in this model, the expected equilibrium price lies strictly above the expected marginal cost (which seems plausible), then we would have obtained the result that the firms have some market power. Hence, in this scenario, Spulber (1995) and the other authors would indeed be right when claiming that asymmetric information softens competition and resolves the Bertrand paradox.

In the present paper I investigate the consequences of relaxing the two assumptions of (i) no drastic innovations and (ii) independent cost draws. I first show, in a fairly general setting, that if the cost draws are to a sufficiently high degree interdependent (i.e., they are "sufficiently common"), then cost uncertainty indeed softens competition—in the sense of, in expected terms, price and profits being higher and consumer surplus being lower. I then study a specification that yields a closed-form solution and show that relaxing assumption (i) while still making assumption (ii) does not alter the result in the earlier literature. Although this specification relies on specific functional forms, it is sufficiently rich to allow for any given number of firms, any level of demand elasticity, and any (positive) level of interdependence between the firms' costs—private costs, common costs, and combinations of those extremes. In addition, the model allows for many different values of the firms' ex ante efficiency level (i.e., a parameter that measures their common likelihood of drawing a relatively low cost signal). The closed-form solution is simple, in pure strategies, and involves price dispersion.[4]

There is a related literature that studies firms' incentive to share information about their own marginal cost parameter under Bertrand competition with *differentiated* goods. Gal-Or (1986) analyzes such a model with two firms and independent cost draws. She shows that not to share information is a dominant strategy for each duopolist. Raith (1996) considers the more general case with n firms and does allow for correlated cost draws. Also Raith, however, assumes differentiated goods. Moreover, neither Gal-Or (1986) nor Raith (1996) makes any comparison between market outcomes with complete and incomplete information. Other related literature includes Milgrom and Weber's (1982) seminal analysis of auctions with affiliated values. Most of their paper concerns the case where bidders are risk neutral and can purchase one unit of a good. However, Section 8 of their paper also considers the case with risk averse bidders, which is similar in spirit to a model with downward-sloping demand. Milgrom and Weber show that, "for models that include both affiliation and risk aversion, the first- and second-price auctions [which, in my setting, correspond to incomplete and complete information] cannot generally be ranked by their expected prices" (p. 1114). The present paper assumes a particular oligopoly setting and then compares the two expected prices, for various values of the parameters that measure demand elasticity and cost correlation. It also makes comparisons of other entities, such as consumer surplus (which has no natural equivalent in the Milgrom-Weber model), across the two informational regimes.

2 Hansen's (1988) results

As explained in the introduction, Spulber (1995) studies a standard one-shot, homogeneous-good Bertrand model with n ex ante identical firms, but adds the assumption that each firm has private information about some characteristic of its production technology—for example, the firm's constant marginal cost. The cost draws in Spulber's model are independent. He shows that in that environment the equilibrium price lies strictly above the marginal cost. Our first goal is to understand whether it is the uncertainty as such that creates market power for the firms in Spulber's model. To that end, it is useful to note that when adding private information about the marginal cost, Spulber makes *two* assumptions:

A1. The firms' marginal cost parameters may differ from each other.

A2a. Each firm has private information about its own marginal cost parameter.

In order to assess the role of asymmetric information, we can compare the outcome of

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[4] Wolfstetter (1999), Lofaro (2002), Abbink and Brandts (2007), and Belleflamme and Peitz (2015) all derive closed-form solutions in a setting with a uniform cost distribution on $[0, 1]$ and linear demand. (In their experimental study, Abbink and Brandts (2007) actually assume a uniform distribution on $[0, 99]$; but this specification is equivalent to the one in the other three papers, although with another scaling of the units in which output and cost are measured.) Wolfstetter (1999) studies a version of the model with a uniform cost distribution on $[0, 1]$ and an inelastic demand and also derives a closed-form solution (this is effectively a standard private-value first-price auction model). The specification yielding a closed-form solution used in the present paper is a substantial generalization of all the above models.

Spulber's model with the outcome of a benchmark that also makes assumption A1 but replaces A2a with:[5]

A2b. The (possibly different) marginal cost parameters are common knowledge among the firms.

Assumptions A1 and A2b give rise to a standard variation of the Bertrand setup, discussed in many textbooks. The equilibrium outcome of that model is that the lowest-cost firm wins the whole market and charges a price equal to the minimum of the monopoly price and the marginal cost of the firm with the second-lowest cost draw. That is, also this model with complete information but cost heterogeneity gives rise to a market price above marginal cost and a positive profit for the lowest-cost firm. This observation raises the question whether the amount of market power in the model A1+A2b is less than that in the model A1+A2a. Put differently, in this environment with independent costs, does asymmetric information soften competition, as Spulber (1995) and the other authors cited in the introduction suggest?

Thanks to work of Hansen (1988), we know that, at least for a special case[6] of Spulber's (1995) model, the answer to the above question is "no." Hansen's model is couched in terms of a procurement auction in which two firms bid for the right to serve a market with a downward-sloping demand, and within that framework he compares the outcomes of an open (descending) auction and a (first-price) sealed bid auction. The open auction is effectively a Bertrand game with complete information (i.e., A1+A2b), whereas the sealed bid auction is the same as Spulber's (1995) incomplete information model (i.e., A1+A2a). Hansen (1988) shows that the sealed bid auction yields a lower expected price than the open auction. It also yields a higher expected total surplus. Under a somewhat stronger assumption about the demand function, Hansen can also show that the sealed bid auction yields a higher expected profit for the firms and a higher expected consumer surplus, meaning that both consumers and firms are better off under incomplete information.

It is instructive to look at what broad arguments Hansen (1988) uses when proving the result that the sealed bid auction yields a lower expected price than the open auction. First he notes that in an open auction the equilibrium price strategies are the same regardless of whether the quantity is variable or fixed (or, in oligopoly language, whether demand is elastic or inelastic). In either case, the lowest-cost firm can win the whole market with a price that equals the marginal cost of the firm with the second-lowest cost. This yields equality (a) in (1):

$$\mathbb{E}\left[p \mid \text{open, variable}\right] \overset{(a)}{=} \mathbb{E}\left[p \mid \text{open, fixed}\right] \overset{(b)}{=} \mathbb{E}\left[p \mid \text{sealed, fixed}\right] \overset{(c)}{>} \mathbb{E}\left[p \mid \text{sealed, variable}\right]. \tag{1}$$

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[5] In fact, Spulber (1995) does compare these two models in Section III of his paper, but not in terms of their competitiveness.

[6] Hansen (1988) assumes that the firms have a constant returns to scale technology and that the uncertainty concerns each firm's marginal cost parameter, which is only one of the possibilities that Spulber (1995) considers. As Spulber, Hansen assumes that the cost draws are independent. Hansen also assumes a duopoly, whereas Spulber allows for an arbitrary number of firms. Finally, Hansen assumes that the support of the unknown marginal cost parameter is such that the monopoly cost always lies above the marginal cost of the second most efficient firm—an assumption that Spulber does not need to make, given the comparisons he makes in his paper.

The argument that yields equality (a) relies critically on Hansen's assumption that the lowest-cost firm's optimal monopoly price always exceeds the marginal cost of the firm with the second-lowest cost (i.e., a "drastic innovation" must not be possible); without that assumption, the equality would be replaced by a "$<$"-sign and Hansen's proof would no longer be valid.

Next, Hansen invokes the revenue equivalence theorem, which in his model says that in a fixed-quantity auction the expected revenue (which equals the expected price) is the same regardless of whether the auction is sealed-bid or open. This is equality (b) in (1). Finally Hansen shows that, in a sealed bid auction, the equilibrium price must be lower when the quantity is variable compared to when it is fixed (inequality (c) in (1)). The intuition for this result is straightforward. In the sealed bid auction, if the firm raises its price, it will have a higher profit if it still wins the market, but the probability of winning has decreased. The optimal price balances those two effects. However, the former (positive) effect is smaller when demand is downward-sloping, as the higher price then leads to a loss of sales. Therefore the expected price must be lower when demand is elastic.

Jointly, the three steps (a), (b), and (c) yield the desired result that the expected price in the sealed bid auction with a variable quantity is lower than the expected price in the open auction with a variable quantity—or, in other words, cost uncertainty in the Bertrand model intensifies competition in the sense that it lowers the expected market price. While step (c) appears to be quite robust, it has already been noted that step (a) relies critically on Hansen's assumption about the support of the cost distribution from which the firms draw their costs. If we allowed for the possibility that the winning firm has such a large cost advantage that it sometimes, under complete information, optimally charges its monopoly price, then the expected price in the open auction with a variable quantity would be lower than the expected price in the open auction with a fixed quantity. This could conceivably also reverse the result that cost uncertainty in the Bertrand model intensifies competition. In the model to be studied in Section 3, the cost advantage of the most efficient firm will indeed sometimes be so large that it corresponds to a drastic innovation.

It is also clear from the above reasoning that Hansen's proof relies on the revenue equivalence theorem (step (b)). In an environment where that theorem does not hold, the proof will not be valid and it is again conceivable that the result could be reversed. One such environment is a common value auction (or, in oligopoly language, a model with common costs). It is known that in such an auction we have $\mathbb{E}\left[p \mid \text{open, fixed}\right] < \mathbb{E}\left[p \mid \text{sealed, fixed}\right]$;[7] that is, equality (b) is replaced by a "$<$"-sign. Whether Hansen's result is reversed would then depend on the relative magnitudes of the inequalities (b) and (c), as they point in different directions. However, if we let the demand elasticity in the model with a variable quantity be very low, we should expect inequality (c) to be very close to an equality, without this affecting the reverse inequality (b). Hence, at least for a sufficiently low demand elasticity and for "sufficiently common" costs, we should expect Hansen's result to be reversed. This reasoning is indeed the logic behind the main result to be presented in the next section.

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[7] See Milgrom and Weber (1982) and Krishna (2002, Ch. 6).

3 The model

There are $n \geq 2$ risk neutral and profit-maximizing firms that compete à la Bertrand in a homogeneous-good product market. The firms are ex ante identical, they choose their prices simultaneously, and they interact only once. Market demand is given by $D(p)$, where p denotes price. This function is twice continuously differentiable and weakly decreasing (i.e., $D'(p) \leq 0$ for all p). The firm that charges the lowest price, denoted by $p_{\min}$, sells the quantity $D(p_{\min})$, while the rival firms sell nothing and make a zero profit. If two or more firms have chosen the same price, they share the market equally.

Each firm has a production technology that is characterized by constant returns to scale, with firm i's marginal cost being denoted by c_i. In particular,

$$c_i = (1-\alpha) s_i + \frac{\alpha}{n} \sum_{j=1}^{n} s_j, \tag{2}$$

where $\alpha \in [0,1]$ is a parameter and s_i is firm i's *signal*. The firms' signals are independently drawn from the cumulative distribution function $F(s_i)$, with support on the unit interval $[0,1]$; this function is twice continuously differentiable and satisfies $F'(s_i) \overset{\text{def}}{=} f(s_i) > 0$ for all $s_i \in (0,1)$. These assumptions imply that, for $\alpha = 0$, the firms' cost parameters are independent; this is the standard framework with *private costs* that is used in Hansen (1988) and Spulber (1995). For $\alpha = 1$ the cost parameters are identical; this is the other polar case: a framework with *common costs*. Values of α between those two extremes capture intermediate situations where the costs are not (fully) common but still interdependent. Given the above notation and assumptions, we can write the profit of a monopolist as $\pi^m(p, c_i) = (p - c_i)D(p)$. Assume that this profit function is strictly quasiconcave in p and let $p^m(c_i)$ denote the optimal monopoly price (i.e., $p^m(c_i) \in \arg\max_p \pi^m(p, c_i)$).

The specification in (2) implies that a firm, when choosing its price, does not know its own cost. The specification also means that firm i's rivals know things about firm i's cost that it does not know itself. Such situations naturally arise in markets where firms, for various reasons, produce and deliver their products or services a significant amount of time after the consumers have made their purchases. An example of this is the market for direct-to-consumer DNA tests.[8] Another example would be newspapers and other companies where the customer purchases a subscription, after which the company delivers the product regularly during an extended period of time (perhaps a year). In such situations, the companies must post their prices prior to knowing about future events that could affect the costs of material, labor, energy, office rents, etc. (examples of such hard-to-predict events could be labor strikes, conflicts leading to oil price fluctuations, changes in tax rates, or pandemics). Any information about such events that is available, is often dispersed among the firms in the market, as different firms tend to get their information from different sources. This characteristic of the market is captured by the model feature that firm i's cost depends on the rivals' signals.

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[8] The testing company must after the purchase mail the test kit to the consumer (perhaps inter-continentally), after which a saliva sample is returned to the company. The sample is then sequenced in a lab that could be capacity constrained, and using a process that involves careful quality checks. For advanced tests, several months of waiting time is common.

I will solve, and then compare, two versions of this model: one where the signals s_i (and thus also the firms' cost parameters) are common knowledge and one where each signal s_i is firm i's private information.

4 Analysis

4.1 Incomplete information

First consider the incomplete information model. That is, assume that each signal s_i is private information of firm i, although it is common knowledge that c_i is given by (2) and that the s_i values are independently drawn from the distribution $F\left(s_i\right)$. Suppose there is a symmetric equilibrium strategy $p^I\left(s\right)$ that is strictly increasing and differentiable in the signal s that the firm observes (the superscript I stands for incomplete information). Denote the inverse of this function by $\chi\left(p\right)$, meaning that χ is the signal value that would give rise to the price p. If firm i expects all its rivals to follow the equilibrium strategy $p^I\left(\cdot\right)$, and it drew a signal s_i and chose a price p_i, then its expected profit is

$$\mathbb{E}[\pi_i \mid s_i] = \left[p_i - \widehat{\Phi}\left(\chi\left(p_i\right), s_i\right)\right] D\left(p_i\right) G\left[\chi\left(p_i\right)\right]. \tag{3}$$

The term $\widehat{\Phi}\left[\chi\left(p_i\right), s_i\right]$ is firm i's expected cost, conditional on the own signal s_i and on being the cheapest firm in the market. In particular, we can compute[9]

$$\widehat{\Phi}\left[\chi\left(p_i\right), s_i\right] \overset{\text{def}}{=} \mathbb{E}[c_i \mid s_i, p_i < p_j \text{ all } j \neq i] = s_i + \frac{\alpha(n-1)}{n}\left[\frac{\int_{\chi(p_i)}^{1} z f(z) dz}{1 - F\left[\chi\left(p_i\right)\right]} - s_i\right]. \tag{4}$$

The last term in (3), $G\left[\chi\left(p_i\right)\right]$, is the probability with which firm i is indeed the cheapest firm in the market and all the rivals thus having drawn signals that exceed $\chi\left(p_i\right)$:

$$G\left[\chi\left(p_i\right)\right] \overset{\text{def}}{=} \left[1 - F\left(\chi\left(p_i\right)\right)\right]^{n-1}. \tag{5}$$

When firm i chooses p_i to maximize (3), it trades off its desire to charge a low price in order to win the market against its desire to set a relatively high price that ensures that it earns a large profit in case it does win the market. At the optimum, the first-order condition $\partial\mathbb{E}[\pi_i \mid s_i]/\partial p_i = 0$ must hold. By rewriting this and by imposing symmetry, we obtain the following differential equation:

$$\frac{\partial p^I(s)}{\partial s} = \frac{\left[(n-1)\, p^I(s) - s - (n-2)\, \Phi\left(s\right)\right] h\left(s\right) D\left(p^I\right)}{D\left(p^I\right) + \left[p^I(s) - \Phi\left(s\right)\right] D'\left(p^I\right)}, \tag{6}$$

where $\Phi\left(s\right) \overset{\text{def}}{=} \widehat{\Phi}(s, s)$ and where $h\left(s\right)$ is the hazard rate associated with the signal distribution, $h\left(s\right) \overset{\text{def}}{=} f\left(s\right) / \left[1 - F\left(s\right)\right]$.

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[9] The proofs of the relationship in (4) and other results that are not shown in the main text can be found in the appendix.

Proposition 1. ***(Equilibrium)*** *There is an equilibrium of the incomplete information model where each firm chooses the price $p^I(s)$ that is defined by the differential equation (6) and the boundary condition $p^I(1) = 1$. For all $s \in [0,1)$, this equilibrium satisfies $p^I(s) > \Phi(s)$ and $\mathbb{E}[\pi_i \mid s] > 0$.*

It follows from the last claim in Proposition 1 that, in the equilibrium under consideration, a firm earns a strictly positive expected profit also from an ex ante point of view (i.e., in expectation prior to having learned its own signal).[10]

4.2 Complete information

Consider now a Bertrand model that is identical to the one above, except that the signals $(s_1, \ldots, s_n)$, and thus also the marginal cost parameters of all the firms, are common knowledge. This is a model that is often analyzed in textbooks; as already discussed in Section 2, the equilibrium outcome is that the firm with the lowest cost draw serves the whole market and charges a price that equals either the second most efficient firm's marginal cost or, if that is lower, the optimal monopoly price:

$$p^C = \min\left\{c_{(2)}, p^m\left[c_{(1)}\right]\right\}. \tag{7}$$

Here, the superscript C is short for complete information, and the subscript (j) indicates the jth order statistic (i.e., the jth lowest realization among the n draws). In general, in this model, the optimal monopoly price may indeed be lower than the nearest rival's marginal cost. However, this cannot happen if demand is inelastic:

$$D(p) = \begin{cases} 1 & \text{if } p \leq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

4.3 Comparison

In this subsection I compare the equilibrium outcomes of the models in subsections 4.1 and 4.2 with regard to market price, consumer surplus, profits, and total surplus. The comparisons are made from an ex ante point of view, that is, in expected terms at the stage where the firms have observed neither their own nor any of their rivals' signals.

Consider first the relatively straightforward case where demand is inelastic. The results for this case are not novel, as they (at least for all intents and purposes) follow from the analysis in Milgrom and Weber (1982). It is nevertheless useful to establish these results in the present framework, before studying the case with elastic demand. Let the subscript (j,m) denote the jth order statistic among m draws, with $m \leq n$ (i.e., we have $s_{(j,n)} = s_{(j)}$).

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[10] The proof of Proposition 1, which can be found in the appendix, is based on the proof of Theorem 2 in Maskin and Riley (2000). In the setting of these authors, the equilibrium is unique. However, Maskin and Riley's uniqueness proof does not obviously carry over to the present setting with a (partially) common cost, and no uniqueness claim is made here. Yet, my conjecture is that the equilibrium in Proposition 1 is the only one that exists.

Proposition 2. ***(Inelastic demand)*** *Suppose demand is inelastic (i.e., $D(p)$ is given by (8)). Then $\mathbb{E}\left[p^C\right] = (1-\alpha\ )\,\mathbb{E}\left[s_{(2)}\right] + \alpha\ \mathbb{E}\left[s\right]$ and*

$$\mathbb{E}\left[p^I\right] = \mathbb{E}\left[p^C\right] + \frac{\alpha}{n-1}\left[\mathbb{E}\left[s_{(2)}\right] - \mathbb{E}\left[s_{(1,n-1)}\right]\right] = \mathbb{E}\left[p^C\right] + \alpha \int_0^1 F(s)\left[1-F(s)\right]^{n-1} ds. \quad (9)$$

Moreover, with incomplete instead of complete information and for all $\alpha \in [0,1]$:

(i) *the expected price is weakly higher ($\mathbb{E}\left[p^I\right] \geq \mathbb{E}\left[p^C\right]$);*

(ii) *the expected consumer surplus is weakly smaller ($\mathbb{E}\left[S^I\right] \leq \mathbb{E}\left[S^C\right]$);*

(iii) *the expected industry profits are weakly higher ($\mathbb{E}\left[\Pi^I\right] \geq \mathbb{E}\left[\Pi^C\right]$);*

(iv) *and the expected total surplus is the same ($\mathbb{E}\left[W^I\right] = \mathbb{E}\left[W^C\right]$).*

The inequalities in (i)-(iii) hold strictly if, and only if, $\alpha \in (0,1]$.

Proposition 2 says that, with private costs, the expected price is the same with complete and with incomplete information (this is the revenue equivalence theorem). However, for any degree of common or partially common costs (so for any $\alpha > 0$), the expected price under incomplete information is strictly higher. That is, in line with Spulber's (1995) intuition as quoted in the Introduction, uncertainty here softens competition. Proposition 2 also says that the consumer is worse off and the firm is better off under incomplete information (with indifference for $\alpha = 0$). This is simply because, with inelastic demand, expected consumer surplus and profits are linear and, respectively, decreasing and increasing in the expected price. Finally, welfare is always the same with complete and with incomplete information. The reason is that the sum of expected consumer surplus and profits does not depend on the expected price, as this cancels out.

Now assume that demand is indeed elastic ($D'(p) < 0$). For this case and with private costs ($\alpha = 0$), Hansen's (1988) analysis told us that private information is pro-competitive ($\mathbb{E}\left[p^I\right] < \mathbb{E}\left[p^C\right]$). The next proposition states that this result is overturned if costs are sufficiently common.

Proposition 3. ***(Sufficiently common cost)*** *Assume that $D'(p) < 0$ for all p. There exists an $\widehat{\alpha} \in [0,1)$ such that, for all $\alpha \in (\widehat{\alpha}, 1]$, with incomplete instead of complete information:*

(i) *the expected price is strictly higher ($\mathbb{E}\left[p^I\right] > \mathbb{E}\left[p^C\right]$);*

(ii) *the expected consumer surplus is strictly smaller ($\mathbb{E}\left[S^I\right] < \mathbb{E}\left[S^C\right]$); and*

(iii) *the expected industry profits are strictly higher ($\mathbb{E}\left[\Pi^I\right] > \mathbb{E}\left[\Pi^C\right]$).*

4.4 Functional Forms that Yield a Closed-Form Solution

To obtain more specific results, I here consider a parametrized version of the model that allows for a closed-form solution. Thus, let the demand function be given by $D(p) = (1-p)^{\epsilon}$, where $\epsilon \geq 0$ is an exogenous parameter. This specification implies that, for a given price p, the parameter ϵ is proportional to the price elasticity of demand: the larger is ϵ, the more elastic is demand (and for $\epsilon = 0$, demand is completely inelastic).[11] Moreover, let the signal distribution be given by

$$F(s_i) = 1 - (1 - s_i)^x, \tag{10}$$

with $x > 0$ being an efficiency parameter: an increase in x makes the firms more efficient from an ex ante perspective, in the sense that this makes lower signal values more likely.[12] I will refer to the version of the model with these two functional forms as the "parametrized version of the model."

4.4.1 Incomplete Information

First consider the parametrized version of the model under incomplete information.

Corollary 1. ***(Closed-form solution)*** *The following is an equilibrium strategy of the parametrized version of the model:*

$$p^I(s_i) = A + (1 - A)s_i, \qquad \textit{where} \qquad A \overset{\textit{def}}{=} \frac{n(1-\alpha)+\alpha}{n[1+\epsilon+(n-1)x]} + \frac{\alpha(n-1)}{n(1+x)}. \tag{11}$$

This version of the model thus has an equilibrium that is quite simple, with each firm charging a price that is linear (or affine) in its own signal.[13] The linear equilibrium strategy satisfies $p^I(1) = 1$, and for all $s_i < 1$ the firm chooses a price strictly above its expected cost conditional on winning, $\Phi(s_i)$. Indeed, in the parametrized version of the model, we can write

$$\Phi(s_i) = C + (1 - C)s_i, \qquad \text{with} \qquad C \overset{\text{def}}{=} \frac{\alpha(n-1)}{n(1+x)}, \quad A > C, \quad \text{and} \quad \lim_{\epsilon \to \infty} A = C. \tag{12}$$

The equilibrium pricing strategy in (11) features all the comparative statics properties that we would expect in this economic environment: the pricing schedule's distance to the expected cost schedule (i.e., $A - C$) is strictly decreasing in the number of firms (n), in the elasticity parameter (ϵ), and in the firms' common level of ex ante efficiency (x). This is reassuring, as it suggests that even if the parametrized version of the model relies on functional forms that are special, in qualitative terms it behaves in the way we would expect a more general model to do. Hence, hopefully also the results to be derived within this framework would hold more generally.[14]

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[11] In particular, the price elasticity of demand equals $\eta(p) \overset{\text{def}}{=} -D'(p)p/D(p) = \epsilon p/(1-p)$.

[12] We can think of each firm i in the market as having access to x research laboratories, each of which produces one independent signal draw from a uniform distribution on $[0, 1]$; the firm then uses the lowest one of the x draws as its signal. This procedure is equivalent to receiving one single draw from the distribution $F(s_i)$, defined in (10) above. Obviously, however, the analysis does not rely on this interpretation or on x being an integer.

[13] As mentioned in the Introduction, this result is a generalization of the models in the existing literature that yield a closed-form solution. By setting $(\alpha, \epsilon, x) = (0, 1, 1)$, we obtain the specifications used in Lofaro (2002), Belleflamme and Peitz (2015), and the elastic-demand model studied by Wolfstetter (1999, p. 237). By setting $(\alpha, \epsilon, x) = (0, 0, 1)$ we obtain the specification in the inelastic-demand model used by Wolfstetter (1999, p. 236).

[14] It is not so clear what comparative statics result to expect in terms of a parameter like α, which captures the strength of the common cost feature (it should depend on the particular specification used to model private/common costs). In the framework that is used here, the equilibrium pricing schedule's distance to the expected cost schedule is strictly decreasing in α: moving closer to fully common costs leads to a smaller price-cost margin.

4.4.2 Complete Information

Next consider the parametrized version of the model under complete information. The optimal price p^C of the firm that wins the market is, again, given by (7), although now we can write

$$p^m\left[c_{(1)}\right]=\frac{1+\varepsilon c_{(1)}}{1+\varepsilon}. \tag{13}$$

The optimal monopoly price in (13) is guaranteed to exceed the second lowest cost parameter (meaning that a drastic innovation is not possible) only if the costs are common to a high enough extent. In particular, we have:

Lemma 1. ***(Possibility of drastic innovations)*** *The relationship* $c_{(2)} \leq p^m\left[c_{(1)}\right]$ *holds for all signal realizations if, and only if,*

$$\alpha \geq \frac{n\varepsilon}{1+n\varepsilon}. \tag{14}$$

4.4.3 Comparison

Comparing the two versions of the model, we obtain the following result.

Proposition 4. *Consider the parametrized version of the model. Assume* $\varepsilon > 0$.

- ***(a)*** ***(Sufficient condition)*** *For all* α *that satisfy (14) and for all* $n \geq 2$, *the expected price is strictly higher with incomplete instead of complete information* ($\mathbb{E}[p^I] > \mathbb{E}[p^C]$).
- ***(b)*** ***(Necessary and sufficient condition)*** *Assume* $n = 2$. *Then there exists a unique* α *such that* $\mathbb{E}[p^I] = \mathbb{E}[p^C]$. *This value of* α, *denoted by* α^*, *satisfies*

$$\alpha^* \in \left(0, \frac{2\varepsilon}{1+x+2\varepsilon}\right), \quad \lim_{\epsilon\to 0}\alpha^* = \lim_{x\to\infty}\alpha^* = 0, \quad \textit{and} \quad \lim_{x\to 0}\alpha^* = \frac{2\varepsilon}{1+2\varepsilon}.$$

Moreover, we have $\mathbb{E}[p^I] < \mathbb{E}[p^C]$ *for all* $\alpha \in [0, \alpha^*)$, *and* $\mathbb{E}[p^I] > \mathbb{E}[p^C]$ *for all* $\alpha \in (\alpha^*, 1]$.

The cutoff value α^* is defined in the proof of the proposition and is graphed in Fig. 1 against the elasticity parameter ϵ, for a few different values of x. For values of α above this cutoff value, the expected price with complete information is lower than with incomplete information.

As argued in the Introduction and in Section 2, the model studied in the present paper could conceivably overturn Hansen's (1988) results also if costs are private (i.e., with $\alpha = 0$). (If this were the case, the reason would be that the signal distribution in the present model allows for a drastic innovation.) To investigate this, I here compare the two models under the assumption of private costs.

Proposition 5. ***(Private costs)*** *Consider the parametrized version of the model. Suppose* $\alpha = 0$ *and* $\epsilon > 0$. *Then, with incomplete information instead of complete information, the expected price is strictly lower* ($\mathbb{E}\left[p^I\right] < \mathbb{E}\left[p^C\right]$).

The relationship reported in Proposition 5 is in line with the one found in Hansen (1988). Thus, at least in the parametrized version of model, Hansen's assumption about the cost

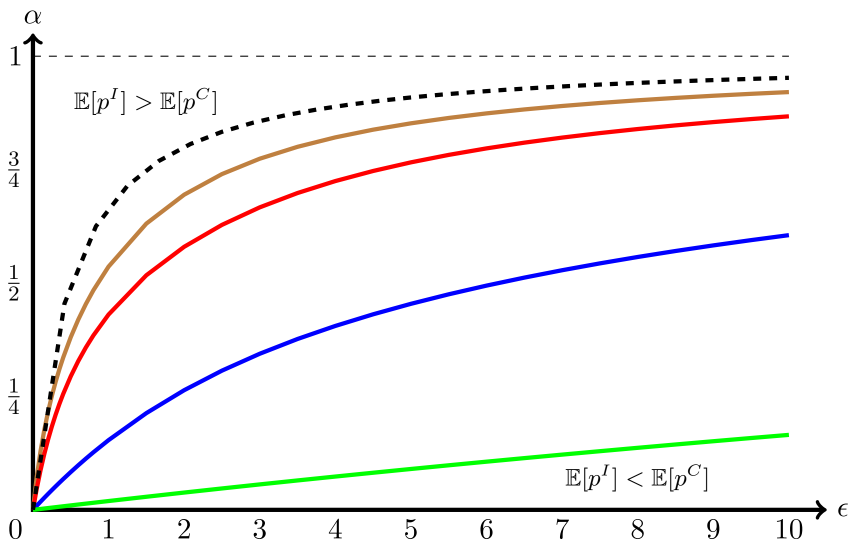

Figure 1: The dashed graph in black shows the sufficient condition stated in part (a) of Proposition 4. The solid graphs in brown, red, blue, and green show the if-and-only-if conditions in part (b) of Proposition 4 for $x = 0.01$, $x = 1$, $x = 10$, and $x = 100$, respectively. All graphs assume $n = 2$.

distribution—implying that the lowest-cost firm's optimal monopoly price always exceeds the second-lowest cost draw—does not matter for the results. The result in Proposition 5 also holds for any arbitrary number of firms, whereas Hansen assumed a duopoly market. We can conclude that, both in Hansen's (1988) model and in the present environment with private costs, asymmetric information is not anti-competitive but pro-competitive, at least in the sense that the firms' equilibrium mark-ups are (in expectation) smaller in an environment with asymmetric information.

5 Conclusions

This paper asked the question whether cost uncertainty softens competition in a homogeneous-good Bertrand model. Previous literature has studied this question in a framework with independent cost draws and found that the answer is "no." In contrast, the present paper showed that when the cost draws are interdependent to a sufficiently large degree, then cost uncertainty indeed softens competition. Specifically, under those conditions the expected price is higher and both expected consumer surplus and total surplus are smaller with incomplete information than with complete information.

The paper also presented a simple and tractable—yet quite rich—parametrized model of price competition with privately known but interdependent cost draws. This model may be useful in applications. For example, it would be interesting, using an infinite-horizon repeated-game version of this framework, to study the effect of cost interdependence on the firms' ability to collude.

Appendix

Preliminaries

The density functions of the first-order and second-order statistics are, respectively, given by

$$f_{s_{(1)}} = nf\left(s_{(1)}\right)\left[1-F\left(s_{(1)}\right)\right]^{n-1} \qquad \text{and} \qquad f_{s_{(2)}} = \frac{n!}{(n-2)!}f\left(s_{(2)}\right)F\left(s_{(2)}\right)\left[1-F\left(s_{(2)}\right)\right]^{n-2}. \tag{15}$$

Similarly, under the assumption that $n=2$, the joint density of the first-order and second-order statistics is

$$2f\left(s_{(1)}\right)f\left(s_{(2)}\right) \qquad \text{if } s_{(1)} \leq s_{(2)} \text{ and 0 otherwise.} \tag{16}$$

The above results can be found in, for example, Gumbel (1958/2004, p. 53), Gut (2009, pp. 102-105), or Wolfstetter (1999, p. 344).

Under the assumption of the parametrized model that $F(s)$ is given by (10), the expected values of a signal s, the first-order statistic $s_{(1)}$, and the second-order statistic $s_{(2)}$ are, respectively, given by

$$\begin{gathered}\mathbb{E}[s] = \int_0^1 sf(s)ds = \frac{1}{1+x}, \qquad \mathbb{E}\left[s_{(1)}\right] = \int_0^1 s_{(1)}f_{s_{(1)}}ds_{(1)} = \frac{1}{1+nx}, \\ \mathbb{E}\left[s_{(2)}\right] = \int_0^1 s_{(2)}f_{s_{(2)}}ds_{(2)} = \frac{1+(2n-1)x}{(1+nx)\left[1+(n-1)x\right]}.\end{gathered} \tag{17}$$

Finally in this subsection with preliminaries, consider the following lemma (to be used in the proof of Proposition 2).

Lemma A1. *We have*

$$\int_0^1 (n-1)f(s)\left[1-F(s)\right]^{n-2}F(s)\frac{\int_s^1 zf(z)\,dz}{1-F(s)}ds = \frac{(n-1)\,\mathbb{E}[s]-\mathbb{E}\left[s_{(1,n-1)}\right]}{(n-1)(n-2)} - \frac{\mathbb{E}\left[s_{(2)}\right]}{n(n-1)}. \tag{18}$$

Proof. First note that, by integrating by parts, we can write

$$\begin{aligned}&\int_0^1 (n-1)f(s)\left[1-F(s)\right]^{n-2}\frac{\int_s^1 zf(z)\,dz}{1-F(s)}ds \\ = \;& -\left[1-F(s)\right]^{n-1}\frac{\int_s^1 zf(z)\,dz}{1-F(s)}\Big|_0^1 + \int_0^1 \left[1-F(s)\right]^{n-1}\frac{-sf(s)\left[1-F(s)\right]+f(s)\int_s^1 zf(z)\,dz}{\left[1-F(s)\right]^2}ds \\ = \;& \mathbb{E}[s] - \int_0^1 sf(s)\left[1-F(s)\right]^{n-2}ds + \int_0^1 f(s)\left[1-F(s)\right]^{n-2}\frac{\int_s^1 zf(z)\,dz}{1-F(s)}ds.\end{aligned}$$

From (15) we have that the density function of $s_{(1)}$ is given by $nf(s)\left[1-F(s)\right]^{n-1}$. Therefore,

$$\int_0^1 sf(s)\left[1-F(s)\right]^{n-2}ds = (n-1)^{-1}\mathbb{E}\left[s_{(1,n-1)}\right],$$

where $\mathbb{E}\left[s_{(1,n-1)}\right]$ is the expected value of the first-order statistic among $n-1$ draws. We can thus equivalently write the above equality as

$$\int_0^1 f(s)\left[1-F(s)\right]^{n-2}\frac{\int_s^1 zf(z)\,dz}{1-F(s)}ds = \frac{\mathbb{E}[s]-(n-1)^{-1}\mathbb{E}\left[s_{(1,n-1)}\right]}{n-2}. \tag{19}$$

Next, integrating by parts yields

$$\begin{aligned}
& \int_0^1 (n-1) f(s) [1-F(s)]^{n-2} F(s) \frac{\int_s^1 z f(z)\, dz}{1-F(s)} ds \\
= \; & -[1-F(s)]^{n-1} F(s) \frac{\int_s^1 z f(z)\, dz}{1-F(s)} \, |_0^1 + \int_0^1 [1-F(s)]^{n-1} h(s) \left[\frac{\int_s^1 z f(z)\, dz}{1-F(s)} - sF(s) \right] ds \\
= \; & \int_0^1 f(s) [1-F(s)]^{n-2} \frac{\int_s^1 z f(z)\, dz}{1-F(s)} ds - \int_0^1 s f(s) F(s) [1-F(s)]^{n-2} ds \\
= \; & \frac{(n-1)\,\mathbb{E}[s] - \mathbb{E}\left[s_{(1,n-1)}\right]}{(n-1)(n-2)} - \frac{E\left[s_{(2)}\right]}{n(n-1)},
\end{aligned}$$

where the last equality uses (19) and the density function of $s_{(2)}$, stated in (15). $\square$

Derivation of a Firm's Expected Cost

Using (2), firm i's cost can be written as $c_i = \left(1 - \alpha + \frac{\alpha}{n}\right) s_i + \frac{\alpha}{n} \sum_{j \neq i} s_j$. From the perspective of a firm that knows that it has the lowest signal, each one of the other firms' signals is distributed according to the density $f(s) / [1 - F[\chi(p_i)]]$, with support $[\chi(p_i), 1]$. Moreover, the other firms' signals are by assumption independent of each other. Thus, a firm's expected cost, conditional on the own signal s_i and on having chosen the lowest price, equals

$$\widehat{\Phi}[\chi(p_i), s_i] \overset{\text{def}}{=} \mathbb{E}[c_i \mid s_i, p_i < p_j \text{ all } j \neq i] = \left(1 - \frac{\alpha(n-1)}{n}\right) s_i + \frac{\alpha(n-1)}{n} \left[\frac{\int_{\chi(p_i)}^1 z f(z) dz}{1 - F[\chi(p_i)]} \right], \tag{20}$$

which simplifies to the expression in (4). Let $\Phi(s)$ denote the function we obtain when evaluating (20) at symmetry:

$$\Phi(s) \overset{\text{def}}{=} \widehat{\Phi}(s, s) = s + \frac{\alpha(n-1)}{n} \left[\frac{\int_s^1 z f(z) dz}{1 - F(s)} - s \right]. \tag{21}$$

Now, differentiate the function in (20) w.r.t. χ:

$$\widehat{\phi}[\chi(p_i)] \overset{\text{def}}{=} \frac{\partial \widehat{\Phi}[\chi(p_i), s_i]}{\partial \chi} = \frac{\alpha(n-1) h[\chi(p_i)]}{n} \left[\frac{\int_{\chi(p_i)}^1 z f(z) dz}{1 - F[\chi(p_i)]} - \chi(p_i) \right]. \tag{22}$$

Let $\phi(s)$ denote the function we obtain when evaluating (22) at symmetry:

$$\phi(s) \overset{\text{def}}{=} \widehat{\phi}(s) = \frac{\alpha(n-1) h(s)}{n} \left[\frac{\int_s^1 z f(z) dz}{1 - F(s)} - s \right] = h(s) [\Phi(s) - s], \tag{23}$$

where the last equality follows from (21).

Proof of Proposition 1

The proof draws on arguments in the proof of Theorem 2 in Maskin and Riley (2000). By the revelation principle, the equilibrium strategy can be represented as a direct mechanism that is incentive compatible and individually rational. Let $\pi(z, s)$ represent the expected profit of a type s firm that acts as a type z firm,

$$\pi(z, s) = \left[p(z) - \widehat{\Phi}(z, s)\right] D[p(z)] G(z). \tag{24}$$

Incentive compatibility requires that $\pi(s) \overset{\text{def}}{=} \pi(s, s) \geq \pi(z, s)$ for all $z \in [0, 1]$. Individual rationality requires that $\pi(s) \geq 0$ for all $s \in [0, 1]$.

Given the boundary condition $p(1) = 1$, we have $\pi(1) = 0$. Moreover, by the envelope theorem, $\pi_2(z, s) = -\widehat{\Phi}_2(z, s) D[p(z)] G(z) < 0$, where we used the fact that $\widehat{\Phi}_2(z, s) > 0$ (cf. (20)). It follows that $\pi(s) \geq 0$ for all

$s \in [0,1]$, thus ensuring that individual rationality holds; in addition, we have $\pi(s) > 0$ and $p(s) > \Phi(s)$ for all $s \in [0,1)$, which were two of the claims in the proposition.

To verify incentive compatibility, assume that $p(z)$ is differentiable and differentiate:

$$\begin{aligned} \frac{\partial \pi(z,s)}{\partial z} &= \left[\left(p'(z) - \frac{\partial \widehat{\Phi}(z,s)}{\partial z}\right) D\left[p(z)\right] + \left(p(z) - \widehat{\Phi}(z,s)\right) D'(p)p'(z)\right] G(z) \\ &\quad + \left[p(z) - \widehat{\Phi}(z,s)\right] D\left[p(z)\right] g(z) \\ &= \left[D(p) + \left(p(z) - \widehat{\Phi}(z,s)\right) D'(p)\right] G(z) \times \left[p'(z) + T(z,s\right], \end{aligned} \tag{25}$$

where

$$T(z,s) \overset{\text{def}}{=} \frac{\left[\left(p(z) - \widehat{\Phi}(z,s)\right) \frac{g(z)}{G(z)} - \frac{\partial \widehat{\Phi}(z,s)}{\partial z}\right] D(p)}{D(p) + \left[p(z) - \widehat{\Phi}(z,s)\right] D'(p)}$$

and $g(z) = G'(z)$. Note that $T(z,s)$ is strictly increasing in s:

$$\frac{\partial T(z,s)}{\partial s} = -\frac{\frac{\partial \widehat{\Phi}(z,s)}{\partial s}\left[D(p)\frac{g(z)}{G(z)} + D'(p)\frac{\partial \widehat{\Phi}(z,s)}{\partial z}\right] D(p)}{\left[D(p) + \left[p(z) - \widehat{\Phi}(z,s)\right] D'(p)\right]^2} > 0, \tag{26}$$

as $\widehat{\Phi}(z,s)$ in strictly increasing in both its arguments (cf. (20) and (22)), $g(z) < 0$, and $D'(p) \leq 0$. Let $p(s)$ be defined by the differential equation with boundary condition

$$p'(s) = -T(s,s) = -\frac{\left[\left(p(s) - \widehat{\Phi}(s,s)\right) \frac{g(s)}{G(s)} - \frac{\partial \widehat{\Phi}(s,s)}{\partial z}\right] D[p(s)]}{D[p(s)] + \left[p(s) - \widehat{\Phi}(s,s)\right] D'[p(s)]}, \qquad p(1) = 1,$$

which simplifies to the differential equation specified in (6). It follows from (25) and (26) that

$$\frac{\partial \pi(z,s)}{\partial z} \gtreqless 0 \text{ as } z \lesseqgtr s.$$

Therefore, $z = s$ yields the global maximum of $\pi(z,s)$ □

Proof of Proposition 2

Under the assumptions of complete information and inelastic demand, the equilibrium price paid by the consumers equals the cost of the second-most efficient firm; that is, $p^C = (1-\alpha)\, s_{(2)} + \frac{\alpha}{n} \sum_{i=1}^{n} s_{(i)}$. Taking expectations yields

$$\mathbb{E}\left[p^C\right] = (1-\alpha)\,\mathbb{E}\left[s_{(2)}\right] + \alpha \mathbb{E}\left[s\right]. \tag{27}$$

For the case with incomplete information and inelastic demand, we can use results from the proof of Proposition 1 but with $D(p) \equiv 1$. Setting the derivative in (25) equal to zero, using $D(p) = 1$ and $D'(p) = 0$, we have

$$\begin{aligned} \frac{\partial \pi(z,s)}{\partial z} &= \left(p'(z) - \frac{\partial \widehat{\Phi}(z,s)}{\partial z}\right) G(z) + \left[p(z) - \widehat{\Phi}(z,s)\right] g(z) = 0 \\ \Leftrightarrow \frac{\partial \left[p(z)\, G(z)\right]}{\partial z} &= \frac{\partial \widehat{\Phi}(z,s)}{\partial z} G(z) + \widehat{\Phi}(z,s)\, g(z). \end{aligned} \tag{28}$$

Evaluating the above first-order condition at the symmetric optimum $(z = s)$, we can write

$$\begin{aligned} \frac{\partial \left[p(s)\, G(s)\right]}{\partial s} &= \phi(s)\, G(s) + \Phi(s)\, g(s) \\ &= h(s)\left[\Phi(s) - s\right] G(s) - (n-1)\Phi(s) G(s)\, h(s) \\ &= -h(s) G(s)\left[s + (n-2)\Phi(s)\right], \end{aligned}$$

where the second equality uses (23) and the relationship $g(s) = -(n-1)h(s)G(s)$. Integrating, we thus obtain

$$p^I(s) = \frac{1}{G(s)} \int_s^1 h(z)G(z)\left[z + (n-2)\Phi(z)\right] dz, \tag{29}$$

where the integration constant has been set equal to zero (this is required by the boundary condition $p(1) = 1$).

We can now compute the expected value of p^I. Using the density of the first-order statistic, we can write

$$\begin{aligned}
\mathbb{E}\left[p^I\right] &= \int_0^1 nf(s)\left[1-F(s)\right]^{n-1} p^I(s)ds = n\int_0^1 f(s)\int_s^1 h(z)G(z)\left[z+(n-2)\Phi(z)\right]dzds \\
&= n\int_0^1 F(s)\,h(s)G(s)\left[s+(n-2)\Phi(s)\right]ds = n\int_0^1 f(s)\,F(s)\left[1-F(s)\right]^{n-2}\left[s+(n-2)\Phi(s)\right]ds \\
&= n\int_0^1 f(s)\,F(s)\left[1-F(s)\right]^{n-2}\left[(n-1)s+\frac{\alpha(n-1)(n-2)}{n}\left[\frac{\int_s^1 zf(z)dz}{1-F(s)}-s\right]\right]ds \\
&= \left(1-\frac{\alpha(n-2)}{n}\right)\mathbb{E}\left[s_{(2)}\right]+\alpha(n-1)(n-2)\int_0^1 f(s)\,F(s)\left[1-F(s)\right]^{n-2}\left[\frac{\int_s^1 zf(z)dz}{1-F(s)}\right]ds \\
&= \left(1-\frac{\alpha(n-2)}{n-1}\right)\mathbb{E}\left[s_{(2)}\right]+\alpha\,\mathbb{E}\left[s\right]-\frac{\alpha}{n-1}\mathbb{E}\left[s_{(1,n-1)}\right] \\
&= (1-\alpha\,)\,\mathbb{E}\left[s_{(2)}\right]+\alpha\,\mathbb{E}\left[s\right]+\frac{\alpha}{n-1}\left[\mathbb{E}\left[s_{(2)}\right]-\mathbb{E}\left[s_{(1,n-1)}\right]\right],
\end{aligned}$$

where the second equality uses (29), the fourth equality uses the definition of $G(s)$, the fifth equality uses the expression for $\Phi(s)$ from (21), and the seventh equality uses Lemma A1. The last line together with (27) yield the first equality in (9). The second equality in (9) follows from noting that, by integrating by parts, we can write

$$\mathbb{E}\left[s_{(2)}\right] = \int_0^1 \left[1-F(s)\right]^n ds + \int_0^1 F(s)\left[1-F(s)\right]^{n-1} ds, \qquad \mathbb{E}\left[s_{(1,n-1)}\right] = \int_0^1 \left[1-F(s)\right]^{n-1} ds.$$

The claim in part (i) of the proposition, about the expected prices, follows immediately from (9). Next consider the claim in (ii) that $\mathbb{E}\left[S^I\right] \leq \mathbb{E}\left[S^C\right]$, with a strict inequality if and only if $\alpha > 0$. But this claim follows immediately from the expected price comparison above and the fact that, with inelastic demand, consumer surplus is given by $S = 1-p$ (so it is linear and decreasing in the price). Similarly, realized industry profits given inelastic demand equal $p - c_{(1)}$ (so they are linear and increasing in the price). This means that also claim (iii) about profits follows immediately from the result about expected price comparison above. Finally, given the expression for consumer surplus and realized profits above, it is clear that welfare (the sum of those terms) is independent of the price. Hence the welfare result, claim (iv), must also hold. ☐

Proof of Proposition 3

From Proposition 1 we know that, for all $s \in (0,1)$ and all $\alpha \in [0,1]$, $p^I(s) > \Phi(s)$. Thus, taking expectations over s, we have $\mathbb{E}\left[p^I(s)\right] > \mathbb{E}\left[\Phi(s)\right]$ for all $\alpha \in [0,1]$. Computing $\mathbb{E}\left[\Phi(s)\right]$, we obtain

$$\begin{aligned}
\mathbb{E}\left[\Phi(s)\right] &= \int_0^1 nf(s)\left[1-F(s)\right]^{n-1}\Phi(s)\,ds = \int_0^1 nf(s)\left[1-F(s)\right]^{n-1}\left[s+\frac{\alpha(n-1)}{n}\left(\frac{\int_s^1 zf(z)dz}{1-F(s)}-s\right)\right]ds \\
&= \left(1-\frac{\alpha(n-1)}{n}\right)\mathbb{E}\left[s_{(1)}\right]+\alpha(n-1)\int_0^1 f(s)\left[1-F(s)\right]^{n-2}\int_s^1 zf(z)\,dzds \\
&= \left(1-\frac{\alpha(n-1)}{n}\right)\mathbb{E}\left[s_{(1)}\right]+\alpha\left[-\left[1-F(s)\right]^{n-1}\int_s^1 zf(z)\,dz\,|_0^1-\int_0^1 sf(s)\left[1-F(s)\right]^{n-1}ds\right] \\
&= \left(1-\frac{\alpha(n-1)}{n}\right)\mathbb{E}\left[s_{(1)}\right]+\alpha\left[\mathbb{E}\left[s\right]-\frac{1}{n}\mathbb{E}\left[s_{(1)}\right]\right] \\
&= (1-\alpha)\,\mathbb{E}\left[s_{(1)}\right]+\alpha\mathbb{E}\left[s\right],
\end{aligned} \tag{30}$$

where the third equality uses integration by parts.

For $\alpha = 1$, all firms have the same cost; thus, for $\alpha = 1$ we have $\mathbb{E}\left[p^C(s)\right] = \mathbb{E}[s] = \mathbb{E}\left[\Phi(s)\,|_{\alpha=1}\right]$, where the last equality follows from (30). It follows that $\mathbb{E}\left[p^I(s)\right] > \mathbb{E}[\Phi(s)]$ for $\alpha = 1$. Moreover, both $\mathbb{E}\left[p^I(s)\right]$ and $\mathbb{E}[\Phi(s)]$ are continuous in α, which means that there exists some $\alpha' < 1$ such that $\mathbb{E}\left[p^I(s)\right] > \mathbb{E}[\Phi(s)]$ for all $\alpha \in [\alpha', 1]$. This establishes the claim made in part (i).

Next turn to the claim made in part (ii). Let $\mathbb{E}_{s_{-i}}$ denote the operator that takes expectations over all $s_j \neq s_i$. For $\alpha = 1$, we can write

$$\mathbb{E}_{s_{-i}}\left[S\left[p^C(s)\right]\right] > S\left[\mathbb{E}_{s_{-i}}\left[p^C(s)\right]\right] = S\left[\mathbb{E}_{s_{-i}}\left[\Phi(s)\,|_{\alpha=1}\right]\right] \geq S\left[p^I(s_i)\,|_{\alpha=1}\right] \quad \text{for all } s_i,$$

where the first inequality follows from strict convexity of S and Jensen's inequality, the equality follows from the results in the proof of part (i), and the last inequality follows from (3) and the fact that the expected equilibrium profits (conditional on s_i) must be non-negative. By taking expectations also over s_i, we obtain $\mathbb{E}\left[S\left[p^C(s)\right]\right] > \mathbb{E}\left[S\left[p^I(s)\right]\right]$ for $\alpha = 1$. Thus, by continuity of the relevant functions, there exists some $\alpha'' < 1$ such that $\mathbb{E}\left[S\left[p^C(s)\right]\right] > \mathbb{E}\left[S\left[p^I(s)\right]\right]$ for all $\alpha \in [\alpha'', 1]$.

Finally turn to the claim in part (iii). For $\alpha = 1$, all firms have the same cost; thus, under complete information, Bertrand competition yields a realized profit of zero for all signals. In contrast, under incomplete information the winning firm's profit, conditional on the own signal, is strictly positive for all $s_i < 1$ (see Proposition 1). Taking expectations, we thus have that $\mathbb{E}\left[\Pi^I\right] > \mathbb{E}\left[\Pi^C\right]$ for $\alpha = 1$. Thus, by continuity of the relevant functions, there exists some $\alpha''' < 1$ such that $\mathbb{E}\left[\Pi^I\right] > \mathbb{E}\left[\Pi^C\right]$ for $\alpha = 1$ for all $\alpha \in [\alpha''', 1]$.

The notation $\widehat{\alpha}$ used in the proposition is defined by $\widehat{\alpha} = \max\{\alpha', \alpha'', \alpha'''\}$. □

Proof of Corollary 1

By making use of the assumed functional forms, we can write

$$\int_s^1 zf(z)dz = s(1-s)^x + \frac{(1-s)^{1+x}}{1+x}$$

and thus

$$\Phi(s) = s + \frac{\alpha(n-1)}{n}\left[\frac{\int_s^1 zf(z)dz}{1-F(s)} - s\right] = s + \frac{\alpha(n-1)}{n}\frac{1-s}{1+x}. \tag{31}$$

Using the assumption $p(s) = A + (1-A)s$, this in turn means that we have

$$p(s) - \Phi(s) = \left[A - \frac{\alpha(n-1)}{n(1+x)}\right](1-s), \tag{32}$$

$$(n-1)p(s) - s - (n-2)\Phi(s) = (n-1)\left[A - \frac{\alpha(n-2)}{n(1+x)}\right](1-s). \tag{33}$$

Evaluating the specified demand function and its derivative at $p(s) = A + (1-A)s$, we obtain

$$D\left[p(s)\right] = (1-A)^{\varepsilon}(1-s)^{\varepsilon} \qquad \text{and} \qquad D'\left[p(s)\right] = -\varepsilon(1-A)^{\varepsilon-1}\,(1-s)^{\varepsilon-1}. \tag{34}$$

Finally note that with the specified functional forms, we have $h(s) = x/(1-s)$ and $p'(s) = 1-A$.

Now plug the expressions above into the differential equation in (6):

$$1 - A = \frac{(n-1)\left[A - \frac{\alpha(n-2)}{n(1+x)}\right](1-s)\frac{x}{1-s}(1-A)^{\varepsilon}(1-s)^{\varepsilon}}{(1-A)^{\varepsilon}(1-s)^{\varepsilon} - \left[A - \frac{\alpha(n-1)}{n(1+x)}\right](1-s)\varepsilon(1-A)^{\varepsilon-1}(1-s)^{\varepsilon-1}}.$$

The root $A = 1$ would not yield a strictly increasing price; thus, we can suppose that $A \neq 1$ and the above equality simplifies to

$$1 = \frac{(n-1)\left[A - \frac{\alpha(n-2)}{n(1+x)}\right]x}{1 - A - \left[A - \frac{\alpha(n-1)}{n(1+x)}\right]\varepsilon} \Leftrightarrow A = \frac{n(1+x) + \alpha\,(n-1)\,[\epsilon + (n-2)x]}{n(1+x)\,[1 + \epsilon + (n-1)\,x]}.$$

The last term can be rewritten as the expression for A stated in (11). $\square$

Proof of Lemma 1

We can write

$$\begin{aligned} c_{(2)} \leq p^m\left[c_{(1)}\right] &\Leftrightarrow \varepsilon\left(c_{(2)} - c_{(1)}\right) \leq 1 - c_{(2)} \\ &\Leftrightarrow \varepsilon(1-\alpha)\left[s_{(2)} - s_{(1)}\right] \leq 1 - (1-\alpha)s_{(2)} - \frac{\alpha}{n}\sum_{j=1}^{n} s_{(j)} \\ &\Leftrightarrow \left[\varepsilon(1-\alpha) - \frac{\alpha}{n}\right]\left[s_{(2)} - s_{(1)}\right] \leq 1 - \left[1 - \frac{(n-1)\alpha}{n}\right]s_{(2)} - \frac{\alpha}{n}\sum_{j=2}^{n} s_{(j)}, \end{aligned} \tag{35}$$

where the first line uses (13) and second line uses the definitions of $c_{(1)}$ and $c_{(2)}$ in (2). If (14) holds, then the left-hand side of the last inequality in (35) is weakly negative. The right-hand side, however, is weakly positive for all realizations of $s_{(j)}$. This proves the "if" part of the claim. The "only if" part follows from noting that if (14) is violated, then the last inequality in (35) fails to hold if $s_{(1)}$ is close enough to zero and if $s_{(j)}$ (for all $j \geq 2$) are close enough to one. $\square$

Proof of Proposition 4

Consider claim (a) in the proposition. From Lemma 1 we know that, for all $\alpha \geq \frac{n\varepsilon}{1+n\varepsilon}$, we have $p^C = c_{(2)}$ and that we therefore can write $\mathbb{E}[p^C] = \mathbb{E}[c_{(2)}] = (1-\alpha)\mathbb{E}[s_{(2)}] + \alpha\mathbb{E}[s]$. Using this expression and (11), we can also, for all $\alpha \geq \frac{n\varepsilon}{1+n\varepsilon}$, write

$$\mathbb{E}[p^I] > \mathbb{E}[p^C] \Leftrightarrow A + (1-A)\mathbb{E}[s_{(1)}] > (1-\alpha)\mathbb{E}[s_{(2)}] + \alpha\mathbb{E}[s] \Leftrightarrow A > \frac{(1-\alpha)\mathbb{E}[s_{(2)}] + \alpha\mathbb{E}[s] - \mathbb{E}[s_{(1)}]}{1 - \mathbb{E}[s_{(1)}]}.$$

By using (11) and (17), the last inequality above can be rewritten as

$$\begin{aligned} &\frac{1}{1+\epsilon+(n-1)\,x} + \frac{\alpha\,(n-1)\,[\epsilon+(n-2)x]}{n\,(1+x)\,[1+\epsilon+(n-1)\,x]} > \frac{1}{1+(n-1)\,x} + \frac{\alpha\left[(n-1)^2\,x - (1+nx)\right]}{n\,(1+x)\,[1+(n-1)\,x]} \Leftrightarrow \\ &\frac{\alpha}{n\,(1+x)}\left[\frac{(n-1)\,[\epsilon+(n-2)x]}{1+\epsilon+(n-1)\,x} - \frac{(n-1)^2\,x - (1+nx)}{1+(n-1)\,x}\right] > \frac{1}{1+(n-1)\,x} - \frac{1}{1+\epsilon+(n-1)\,x}. \end{aligned} \tag{36}$$

One can show that the expression in large square brackets on the left-hand side of (36) is strictly positive (it is increasing in ϵ and strictly positive for $\epsilon = 0$). Hence, (36) holds for all $\alpha \geq n\epsilon/(1+n\epsilon)$ if it holds when evaluated at $\alpha = n\epsilon/(1+n\epsilon)$. Plugging $\alpha = n\epsilon/(1+n\epsilon)$ into (36) and rewriting the right-hand side, we have

$$\begin{aligned} &\frac{\epsilon}{(1+n\epsilon)\,(1+x)}\left[\frac{(n-1)\,[\epsilon+(n-2)x]}{1+\epsilon+(n-1)\,x} - \frac{(n-1)^2\,x - (1+nx)}{1+(n-1)\,x}\right] > \frac{\epsilon}{[1+(n-1)\,x]\,[1+\epsilon+(n-1)\,x]} \Leftrightarrow \\ &(n-1)\,[\epsilon+(n-2)x]\,[1+(n-1)\,x] - \left[(n-1)^2\,x - (1+nx)\right][1+\epsilon+(n-1)\,x] > (1+n\epsilon)\,(1+x)\,. \end{aligned}$$

The left-hand side of the last inequality can be written as $(1+x)\,[1+(n-1)x+n\epsilon]$, which means that the term $(1+x)$ cancels out and it is then easy to see that the inequality always holds. This establishes part (a) of the proposition.

Now turn to the claims in part (b) of the proposition. All those claims, except for the one about the limit value, follow if we can show that, given $n = 2$, $\mathbb{E}[p^I]$ and $\mathbb{E}[p^C]$ have the following four properties:

(i) At $\alpha = 0$, $\mathbb{E}[p^I] < \mathbb{E}[p^C]$.

(ii) For all $\alpha \in (0,1)$, $\mathbb{E}[p^I]$ is linear and strictly increasing in α.

(iii) For all $\alpha \in \left[\frac{2\varepsilon}{1+2\varepsilon}, 1\right]$, $\mathbb{E}[p^I] > \mathbb{E}[p^C]$.

(iv) For all $\alpha \in \left(0, \frac{2\varepsilon}{1+2\varepsilon}\right)$, $\mathbb{E}[p^C]$ is strictly concave in α.

Property (i) is shown in the proof of Proposition 5 below. Similarly, property (iii) is implied by the (a) part of Proposition 4, which was proven above. To prove property (ii), first note that it follows immediately from (11) that $\mathbb{E}[p^I]$ is a linear (affine) function of α. Moreover we have

$$\frac{\partial A}{\partial \alpha} = -\frac{n-1}{n[1+\varepsilon+(n-1)x]} + \frac{n-1}{n(1+x)},$$

which by inspection is strictly positive for all $\varepsilon > 0$ and all $n \geq 2$.

Finally, to prove property (iv) first note that, with $n = 2$, eq. (35) yields

$$c_{(2)} \leq p^m\left[c_{(1)}\right] \overset{\text{def}}{=} \frac{1+\varepsilon c_{(1)}}{1+\varepsilon} \Leftrightarrow s_{(2)} \leq \varphi\left[s_{(1)}\right] \overset{\text{def}}{=} \frac{1+\left[\epsilon\left(1-\alpha\right)-\frac{\alpha}{2}\right]s_{(1)}}{\left(1+\epsilon\right)\left(1-\alpha\right)+\frac{\alpha}{2}}.$$

The joint density function of $s_{(1)}$ and $s_{(2)}$ is stated in (16). Using this we can, for $n = 2$ and $\alpha < \frac{2\varepsilon}{1+2\varepsilon}$, write

$$\mathbb{E}\left[p^C\right] = 2\int_0^1 f\left[s_{(1)}\right]\left[\int_{s_{(1)}}^{\varphi\left[s_{(1)}\right]} c_{(2)} f\left[s_{(2)}\right] ds_{(2)} + \int_{\varphi\left[s_{(1)}\right]}^1 p^m\left[c_{(1)}\right] f\left[s_{(2)}\right] ds_{(2)}\right] ds_{(1)}.$$

Differentiating with respect to α yields

$$\frac{\partial \mathbb{E}\left[p^C\right]}{\partial \alpha} = 2\int_0^1 f\left[s_{(1)}\right]\int_{\varphi\left[s_{(1)}\right]}^1 \frac{\partial p^m\left[c_{(1)}\right]}{\partial \alpha} f\left[s_{(2)}\right] ds_{(2)} ds_{(1)}, \tag{37}$$

where

$$\frac{\partial p^m\left[c_{(1)}\right]}{\partial \alpha} = \frac{\epsilon\left[s_{(2)} - s_{(1)}\right]}{2\left(1+\epsilon\right)}.$$

Note that, in (37), α appears only in the expression for $\varphi\left[s_{(1)}\right]$. Thus, differentiating a second time with respect to α yields

$$\frac{\partial^2 \mathbb{E}\left[p^C\right]}{\partial \alpha^2} = -\int_0^1 \frac{\epsilon\left[\varphi\left(s_{(1)}\right) - s_{(1)}\right]}{1+\epsilon} f\left[s_{(1)}\right] f\left[\varphi\left(s_{(1)}\right)\right] ds_{(1)} < 0,$$

which yields the desired result.

It remains to show the claim that $\lim_{\epsilon \to 0} \alpha^* = 0$. However, this follows from the result that $\alpha^* < n\epsilon/(1+n\epsilon)$, in conjunction with fact that $\alpha^* \geq 0$. $\square$

Proof of Proposition 5

For later reference, first note that, for an arbitrary n and with $\alpha = 0$, we can write

$$\mathbb{E}\left[p^C\right] = \int_0^1 \left[\int_{s_{(1)}}^{p^m\left[s_{(1)}\right]} s_{(2)} k\left[s_{(1)}, s_{(2)}\right] ds_{(2)} + \int_{p^m\left[s_{(1)}\right]}^1 p^m\left[s_{(1)}\right] k\left[s_{(1)}, s_{(2)}\right] ds_{(2)}\right] ds_{(1)}, \tag{38}$$

where

$$p^m\left[s_{(1)}\right] = \frac{1+\epsilon s_{(1)}}{1+\epsilon} \qquad \text{and} \qquad k\left[s_{(1)}, s_{(2)}\right] = n(n-1) f\left[s_{(1)}\right] f\left[s_{(2)}\right]\left[1 - F\left(s_{(2)}\right)\right]^{n-2}$$

if $s_{(1)} \leq s_{(2)}$ and $k\left[s_{(1)}, s_{(2)}\right] = 0$ otherwise. Differentiating w.r.t. ϵ, we have

$$\frac{\partial \mathbb{E}\left[p^C\right]}{\partial \epsilon} = \int_0^1 \int_{p^m\left[s_{(1)}\right]}^1 \frac{\partial p^m\left[s_{(1)}\right]}{\partial \epsilon} k\left[s_{(1)}, s_{(2)}\right] ds_{(2)} ds_{(1)},$$

where

$$\frac{\partial p^m\left[s_{(1)}\right]}{\partial \epsilon} = -\frac{1 - s_{(1)}}{\left(1+\epsilon\right)^2}.$$

Using the functional forms for the distributions, we get

$$\begin{aligned}
\frac{\partial \mathbb{E}\left[p^C\right]}{\partial \epsilon} &= -\int_0^1 \left[n(n-1)x^2 \int_{\frac{1+\epsilon s_{(1)}}{1+\epsilon}}^1 \frac{1-s_{(1)}}{(1+\epsilon)^2} \left[1-s_{(1)}\right]^{x-1} \left[1-s_{(2)}\right]^{x-1} \left[1-s_{(2)}\right]^{x(n-2)} ds_{(2)}\right] ds_{(1)} \\
&= -\frac{n(n-1)x^2}{(1+\epsilon)^2} \int_0^1 \left[1-s_{(1)}\right]^x \left[\int_{\frac{1+\epsilon s_{(1)}}{1+\epsilon}}^1 \left[1-s_{(2)}\right]^{x(n-1)-1} ds_{(2)}\right] ds_{(1)} \\
&= \frac{nx}{(1+\epsilon)^2} \int_0^1 \left[1-s_{(1)}\right]^x \left[\left[1-s_{(2)}\right]^{x(n-1)} |_{\frac{1+\epsilon s_{(1)}}{1+\epsilon}}^1\right] ds_{(1)} \\
&= -\frac{nx}{(1+\epsilon)^2} \int_0^1 \left[1-s_{(1)}\right]^x \left[1-\frac{1+\epsilon s_{(1)}}{1+\epsilon}\right]^{x(n-1)} ds_{(1)} \\
&= -\frac{nx\epsilon^{x(n-1)}}{(1+\epsilon)^{2+x(n-1)}} \int_0^1 \left[1-s_{(1)}\right]^{xn} ds_{(1)} \\
&= \frac{nx\epsilon^{x(n-1)}}{(nx+1)\,(1+\epsilon)^{2+x(n-1)}} \left[1-s_{(1)}\right]^{nx+1} |_0^1 = -\frac{nx\epsilon^{x(n-1)}}{(nx+1)\,(1+\epsilon)^{2+x(n-1)}}. \qquad (39)
\end{aligned}$$

Differentiating a second time w.r.t. ϵ, we have

$$\frac{\partial^2 \mathbb{E}\left[p^C\right]}{\partial \epsilon^2} = -\frac{nx\epsilon^{x(n-1)-1}\left[x(n-1)-2\epsilon\right]}{(nx+1)\,(1+\epsilon)^{3+x(n-1)}}. \qquad (40)$$

Moreover, by (11) and (17) and with $\alpha = 0$, the expected price under incomplete information equals

$$\begin{aligned}
\mathbb{E}\left[p^I\right] &= A + (1-A)\,\mathbb{E}\left[s_{(1)}\right] = \frac{1}{1+(n-1)\,x+\epsilon} + \left[1-\frac{1}{1+(n-1)\,x+\epsilon}\right]\frac{1}{1+nx} \\
&= \frac{1}{1+nx} + \frac{1}{1+(n-1)\,x+\epsilon}\left[1-\frac{1}{1+nx}\right] = \frac{1}{1+nx} + \frac{nx}{\left[1+(n-1)\,x+\epsilon\right](1+nx)}. \qquad (41)
\end{aligned}$$

We thus have

$$\frac{\partial \mathbb{E}\left[p^I\right]}{\partial \epsilon} = -\frac{nx}{\left[1+(n-1)\,x+\epsilon\right]^2(1+nx)}, \qquad \frac{\partial^2 \mathbb{E}\left[p^I\right]}{\partial \epsilon^2} = \frac{2nx}{\left[1+(n-1)\,x+\epsilon\right]^3(1+nx)}. \qquad (42)$$

Now turn to the claim in the proposition. Let Δ denote the extent to which $\mathbb{E}\left[p^C\right]$ is larger than $\mathbb{E}\left[p^I\right]$:

$$\Delta(\epsilon, x, n) = \mathbb{E}\left[p^C\right] - \mathbb{E}\left[p^I\right].$$

The claim in the proposition will follow if we can show that $\Delta(\epsilon, x, n)$ has the following three properties:

(i) It equals zero at $\epsilon = 0$: $\Delta(0, x, n) = 0$.

(ii) In the limit as ϵ approaches infinity, it equals zero: $\lim_{\epsilon\to\infty} \Delta(\epsilon, x, n) = 0$.

(iii) For all $\epsilon \in [1+(n-1)\,x, \infty)$, $\Delta(\epsilon, x, n)$ is decreasing in ϵ.

(iv) For all $\epsilon \in (0, 1+(n-1)\,x)$, $\Delta(\epsilon, x, n)$ is quasiconcave in ϵ.

Property (i) follows immediately from (9). The fact that property (ii) holds follows from noting, from (11) and (38), that both $\mathbb{E}\left[p^I\right]$ and $\mathbb{E}\left[p^C\right]$ approach $\mathbb{E}[s_{(1)}]$ as $\epsilon \to \infty$. Next, consider property (iii). By using (39) and (42), we can write

$$\begin{aligned}
\frac{\partial \Delta(\epsilon, x, n)}{\partial \epsilon} \leq 0 \Leftrightarrow -\frac{nx\epsilon^{x(n-1)}}{(nx+1)\,(1+\epsilon)^{2+x(n-1)}} &\leq -\frac{nx}{\left[1+(n-1)\,x+\epsilon\right]^2(1+nx)} \\
\Leftrightarrow \epsilon^{x(n-1)}\left[1+(n-1)\,x+\epsilon\right]^2 &\geq (1+\epsilon)^{2+x(n-1)} \qquad (43) \\
\Leftrightarrow x(n-1)\ln(\epsilon) + 2\ln\left[1+(n-1)\,x+\epsilon\right] - \left[2+x(n-1)\right]\ln(1+\epsilon) &\overset{\text{def}}{=} \psi(\epsilon, x, n) \geq 0
\end{aligned}$$

By evaluating this expression at $\epsilon = 1 + (n-1)x$, we obtain

$$\psi[1 + (n-1)x, x, n] = \ln(4) + [2 + (n-1)x] \ln \left[\frac{1 + (n-1)\,x}{2 + (n-1)\,x} \right], \tag{44}$$

which can be shown to be strictly positive (the expression in (44) equals zero when evaluated at $(n-1)x = 0$, it approaches $\ln(4) - 1$ as $(n-1)x$ approaches infinity, and it is is strictly concave in $(n-1)x$). Moreover, $\psi(\epsilon, x, n)$ can be shown to approach zero as as ϵ approaches infinity and, for all $\epsilon \geq 1 + (n-1)\,x$, to be decreasing in ϵ. It follows that $\Delta(\epsilon, x, n)$ is decreasing in ϵ for all $\epsilon > 1 + (n-1)\,x$.

Finally, consider property (iv). By using (40) and (42), we can write

$$\frac{\partial^2 \Delta(\epsilon, x, n)}{\partial \epsilon^2} = -\frac{nx\epsilon^{x(n-1)-1}\,[x(n-1) - 2\epsilon]}{(nx+1)\,(1+\epsilon)^{3+x(n-1)}} - \frac{2nx}{[1 + (n-1)\,x + \epsilon]^3\,(1+nx)}.$$

Hence,

$$\frac{\partial^2 \Delta(\epsilon, x, n)}{\partial \epsilon^2} \leq 0 \Leftrightarrow -\frac{\epsilon^{x(n-1)-1}\,[x(n-1) - 2\epsilon]}{(1+\epsilon)^{3+x(n-1)}} \leq \frac{2}{[1 + (n-1)\,x + \epsilon]^3}.$$

Evaluating at an ϵ that satisfies $\frac{\partial \Delta(\epsilon, x, n)}{\partial \epsilon} = 0$, thus using (43), we obtain

$$\begin{aligned}
-\frac{\epsilon^{x(n-1)-1}\,[x(n-1) - 2\epsilon]}{(1+\epsilon)\,\epsilon^{x(n-1)}\,[1 + (n-1)\,x + \epsilon]^2} &\leq \frac{2}{[1 + (n-1)\,x + \epsilon]^3} \\
\Leftrightarrow -[x(n-1) - 2\epsilon]\,[1 + (n-1)\,x + \epsilon] &\leq 2\epsilon(1+\epsilon) \\
\Leftrightarrow 2\epsilon(n-1)x \leq x(n-1)\,[1 + (n-1)\,x + \epsilon] &\Leftrightarrow \epsilon \leq 1 + (n-1)\,x.
\end{aligned}$$

That is, for all $\epsilon \in (0, 1 + (n-1)\,x)$, $\Delta(\epsilon, x, n)$ is quasiconcave in ϵ. ☐

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