Supplementary Material to "Does Cost Uncertainty in the Bertrand Model Soften Competition?"

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Abstract

In this note I provide derivations of some results that were not proven in Lagerlöf (2013).

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1 Introduction

In this note I provide derivations of some results that were not proven in Lagerlöf (2013).

2 Derivations of results for the complete information model

Here I derive the expressions that are reported in Table 1 of Lagerlöf (2013), the column for the complete information model. I consider, in turn, expected price, expected industry profits, expected consumer surplus, and expected total surplus. First, however, I state and prove some intermediate results that I need to derive the expressions in Table 1.

2.1 Preliminaries

The probability density function of the lowest draw, $c_{(1)}$, is given by[1]

$$g_1\left[c_{(1)}\right] = n\left[1 - F\left[c_{(1)}\right]\right]^{n-1} f\left[c_{(1)}\right] = nx\left[1 - c_{(1)}\right]^{nx-1},$$

where

$$F(c_i) = 1 - (1 - c_i)^x$$

and

$$f(c_i) = x(1 - c_i)^{x-1}.$$

The expected value of $c_{(1)}$ is therefore given by

$$\begin{aligned} E\left[c_{(1)}\right] &= \int_0^1 c_{(1)} g_1\left[c_{(1)}\right] dc_{(1)} = \int_0^1 c_{(1)} n\left[1 - F\left[c_{(1)}\right]\right]^{n-1} f\left[c_{(1)}\right] dc_{(1)} \\ &= -\left[1 - F\left[c_{(1)}\right]\right]^n c_{(1)} \mid_0^1 + \int_0^1 \left[1 - F\left[c_{(1)}\right]\right]^n dc_{(1)} \\ &= \int_0^1 \left[1 - F\left[c_{(1)}\right]\right]^n dc_{(1)}, \end{aligned}$$

where the third equality is obtained by integrating by parts. Now, using the specific functional form that is assumed, we have

$$E\left[c_{(1)}\right] = \int_0^1 \left[1 - c_{(1)}\right]^{nx} dc_{(1)} = \frac{1}{1 + nx}. \tag{1}$$

Recall from Lagerlöf (2013) that the market price (i.e., the price charged by the firm with the lowest cost draw) equals

$$p^{II} = A + (1 - A) c_{(1)}, \qquad \text{with } A = \frac{1}{(n-1)x + 2}. \tag{2}$$

[1] In order to derive this expression for g_1, let G_1 be the cumulative distribution function associated with the density g_1. The probability that n independent draws all yield a cost equal to c or higher is given by $1 - G_1(c) = [1 - F(c)]^n$, so $G_1(c) = 1 - [1 - F(c)]^n$ and $g_1(c) = n[1 - F(c)]^{n-1} f(c)$.

Also recall from Lagerlöf (2013) that the equilibrium market price under complete information is

$$p^{CI} = \min\left\{c_{(2)}, \frac{1 + c_{(1)}}{2}\right\}$$

and that the joint probability density function of $c_{(1)}$ and $c_{(2)}$ is given by [see, for example, Gumbel (1958/2004, p. 53)]

$$g_{1,2}\left[c_{(1)}, c_{(2)}\right] = n(n-1) f\left[c_{(1)}\right] f\left[c_{(2)}\right]\left[1 - F\left(c_{(2)}\right)\right]^{n-2} \tag{3}$$

if $c_{(1)} \leq c_{(2)}$ and 0 otherwise.

Lemma 1 *For any $a > 0$ we have*

$$\int_0^1 (1-c)^a \, dc = \frac{1}{1+a}.$$

Proof: Straightforward and hence omitted.

Lemma 2 *For any $a > 0$ we have*

$$\int_0^1 c(1-c)^a \, dc = \frac{1}{(1+a)(2+a)}. \tag{4}$$

Proof: By differentiating the right-hand side, one can verify that the following holds:

$$\int c(1-c)^a \, dc = -(1-c)^{a+1} \frac{(a+1)c + 1}{(1+a)(2+a)}. \tag{5}$$

Given (5), (4) follows immediately. □

Lemma 3 *For any $a > 0$ and any $b \in [0, 1]$ we have*

$$\int_{\frac{1+b}{2}}^1 (1-c)^a \, dc = \frac{\left(\frac{1-b}{2}\right)^{a+1}}{1+a}. \tag{6}$$

Proof:

$$\int_{\frac{1+b}{2}}^1 (1-c)^a \, dc = \frac{1}{1+a}\left[-(1-c)^{a+1}\right]_{\frac{1+b}{2}}^1 = \frac{\left(1 - \frac{1+b}{2}\right)^{a+1}}{1+a},$$

which simplifies to the expression in (6).

Lemma 4 *For any $a > 0$ and any $b \in [0, 1]$ we have*

$$\begin{aligned}\int_b^{\frac{1+b}{2}} c(1-c)^a \, dc &= \frac{b(1-b)^{1+a}\left[1 - \left(\frac{1}{2}\right)^{2+a}\right]}{2+a} \\ &\quad + \frac{(1-b)^{1+a}\left[1 - \left(\frac{1}{2}\right)^{2+a}(3+a)\right]}{(1+a)(2+a)}.\end{aligned} \tag{7}$$

Proof: Using (5) we can write

$$\begin{aligned}
&\int_b^{\frac{1+b}{2}} c\,(1-c)^a\,dc \\
= \;& \left[-(1-c)^{a+1}\,\frac{(a+1)\,c+1}{(1+a)\,(2+a)}\right]_b^{\frac{1+b}{2}} \\
= \;& (1-b)^{a+1}\,\frac{(a+1)\,b+1}{(1+a)\,(2+a)} - \left(\frac{1-b}{2}\right)^{a+1}\frac{(a+1)\,\frac{1+b}{2}+1}{(1+a)\,(2+a)} \\
= \;& \frac{(1-b)^{a+1}}{(1+a)\,(2+a)}\left[(a+1)\,b+1-\left(\frac{1}{2}\right)^{a+2}\left[(a+1)\,(1+b)+2\right]\right],
\end{aligned}$$

which simplifies to (7). □

Lemma 5 *For any $a > 0$ and any $b \in [0,1]$ we have*

$$\int_b^{\frac{1+b}{2}} (1-c)^a\,dc = \frac{(1-b)^{a+1}\left[1-\left(\frac{1}{2}\right)^{a+1}\right]}{1+a}. \tag{8}$$

Proof:

$$\begin{aligned}
\int_b^{\frac{1+b}{2}} (1-c)^a\,dc &= \left[-\frac{(1-c)^{a+1}}{1+a}\right]_b^{\frac{1+b}{2}} = \frac{(1-b)^{a+1}}{1+a} - \frac{\left(1-\frac{1+b}{2}\right)^{a+1}}{1+a} \\
&= \frac{(1-b)^{a+1}}{1+a} - \frac{(1-b)^{a+1}\left(\frac{1}{2}\right)^{a+1}}{1+a},
\end{aligned}$$

which simplifies to (8). □

2.2 Expected price

The expected price is

$$\begin{aligned}
E\left[p^{CI}\right] &= \underbrace{\int_0^1 \int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}} c_{(2)} g_{1,2}\left[c_{(1)}, c_{(2)}\right] dc_{(2)} dc_{(1)}}_{=D_1} \\
&\quad \underbrace{+\int_0^1 \int_{\frac{1+c_{(1)}}{2}}^1 \frac{1+c_{(1)}}{2} g_{1,2}\left[c_{(1)}, c_{(2)}\right] dc_{(2)} dc_{(1)}.}_{=D_2}
\end{aligned} \tag{9}$$

Using (3), we can write the first term of (9), D_1, as

$$D_1 = n\,(n-1)\int_0^1 f\left[c_{(1)}\right]\left[\int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}} c_{(2)} f\left[c_{(2)}\right]\left[1-F\left(c_{(2)}\right)\right]^{n-2} dc_{(2)}\right] dc_{(1)}. \tag{10}$$

The "inside integral" of (10) becomes

$$\begin{aligned}
&\int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}} c_{(2)} f\left[c_{(2)}\right]\left[1-F\left(c_{(2)}\right)\right]^{n-2} dc_{(2)} \\
=\; & x \int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}} c_{(2)}\left(1-c_{(2)}\right)^{x-1+(n-2)x} dc_{(2)} \\
=\; & x \frac{c_{(1)}\left(1-c_{(1)}\right)^{(n-1)x}\left[1-\left(\frac{1}{2}\right)^{(n-1)x+1}\right]}{(n-1)x+1} + x \frac{\left(1-c_{(1)}\right)^{(n-1)x}\left[1-\left(\frac{1}{2}\right)^{(n-1)x+1}\left[(n-1)x+2\right]\right]}{(n-1)x\left[(n-1)x+1\right]},
\end{aligned}$$

where the last equality uses Lemma 4. Plugging this expression back into (10) one has

$$\begin{aligned}
& D_1 \\
=\; & \frac{n(n-1)x^2\left[1-\left(\frac{1}{2}\right)^{(n-1)x+1}\right]}{(n-1)x+1} \int_0^1 c_{(1)}\left(1-c_{(1)}\right)^{nx-1} \\
& + \frac{n(n-1)x^2\left[1-\left(\frac{1}{2}\right)^{(n-1)x+1}\left[(n-1)x+2\right]\right]}{(n-1)x\left[(n-1)x+1\right]} \int_0^1 \left(1-c_{(1)}\right)^{nx-1} dc_{(1)} \\
=\; & \frac{(n-1)x\left[1-\left(\frac{1}{2}\right)^{(n-1)x+1}\right]}{(nx+1)\left[(n-1)x+1\right]} + \frac{\left[1-\left(\frac{1}{2}\right)^{(n-1)x+1}\left[(n-1)x+2\right]\right]}{\left[(n-1)x+1\right]} \\
=\; & \frac{(n-1)x+(nx+1)}{(nx+1)\left[(n-1)x+1\right]} \\
& - \frac{\left\{(n-1)x+(nx+1)\left[(n-1)x+2\right]\right\}\left(\frac{1}{2}\right)^{(n-1)x+1}}{(nx+1)\left[(n-1)x+1\right]}, \qquad (11)
\end{aligned}$$

where the second equality uses Lemmas 1 and 2. Using (3), we can write the second term of (9), D_2, as

$$\begin{aligned}
D_2 &= n(n-1)\int_0^1 \frac{1+c_{(1)}}{2} f\left[c_{(1)}\right]\left[\int_{\frac{1+c_{(1)}}{2}}^1 f\left[c_{(2)}\right]\left[1-F\left(c_{(2)}\right)\right]^{n-2} dc_{(2)}\right] dc_{(1)} \\
&= n(n-1)x^2 \int_0^1 \frac{1+c_{(1)}}{2}\left(1-c_{(1)}\right)^{x-1}\left[\int_{\frac{1+c_{(1)}}{2}}^1 \left(1-c_{(2)}\right)^{x-1+x(n-2)} dc_{(2)}\right] dc_{(1)} \\
&= \frac{n(n-1)x^2}{2}\int_0^1 \left(1+c_{(1)}\right)\left(1-c_{(1)}\right)^{x-1}\left[\frac{\left(\frac{1-c_{(1)}}{2}\right)^{x(n-1)}}{x(n-1)}\right] dc_{(1)} \\
&= nx\left(\frac{1}{2}\right)^{x(n-1)+1}\left[\int_0^1 \left(1-c_{(1)}\right)^{xn-1} dc_{(1)} + \int_0^1 c_{(1)}\left(1-c_{(1)}\right)^{xn-1} dc_{(1)}\right] \\
&= nx\left(\frac{1}{2}\right)^{x(n-1)+1}\left[\frac{1}{nx}+\frac{1}{nx(nx+1)}\right] = \frac{(nx+2)\left(\frac{1}{2}\right)^{x(n-1)+1}}{nx+1}, \qquad (12)
\end{aligned}$$

where the third equality uses Lemma 3 and the last equality uses Lemmas 1 and 2.

So summing up, using (11) and (12), we have

$$E\left[p^{CI}\right] = \overbrace{\frac{(n-1)x+(nx+1)}{(nx+1)[(n-1)x+1]} - \frac{\{(n-1)x+(nx+1)[(n-1)x+2]\}\left(\frac{1}{2}\right)^{(n-1)x+1}}{(nx+1)[(n-1)x+1]}}^{\text{from (11)}}$$
$$+\overbrace{\frac{(nx+2)[(n-1)x+1]\left(\frac{1}{2}\right)^{x(n-1)+1}}{(nx+1)[(n-1)x+1]}}^{\text{from (12)}}.$$

The numerator terms multiplied by $\left(\frac{1}{2}\right)^{(n-1)x+1}$ reduce to the following:

$$\begin{aligned} & (nx+2)[(n-1)x+1] - \{(n-1)x+(nx+1)[(n-1)x+2]\} \\ = \; & (nx+2)[(n-1)x+1] - (n-1)x - (nx+1)[(n-1)x+1] - (nx+1) \\ = \; & [(n-1)x+1] - (n-1)x - (nx+1) = -nx. \end{aligned}$$

Therefore,

$$E\left[p^{CI}\right] = \frac{(2n-1)x+1}{(nx+1)[(n-1)x+1]} - \frac{nx\left(\frac{1}{2}\right)^{(n-1)x+1}}{(nx+1)[(n-1)x+1]},$$

which is identical to the expression in Table 1.

2.3 Expected industry profits

The expected industry profits are

$$E\left[\Pi^{CI}\right] = \overbrace{\int_0^1 \int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}} \left[1-c_{(2)}\right]\left[c_{(2)}-c_{(1)}\right] g_{1,2}\left[c_{(1)},c_{(2)}\right] dc_{(2)}dc_{(1)}}^{=K_1}$$
$$\underbrace{+\int_0^1 \int_{\frac{1+c_{(1)}}{2}}^{1} \left[1-\frac{1+c_{(1)}}{2}\right]\left[\frac{1+c_{(1)}}{2}-c_{(1)}\right] g_{1,2}\left[c_{(1)},c_{(2)}\right] dc_{(2)}dc_{(1)}}_{=K_2}.$$

Rewrite the first term, K_1, using (3):

$$\begin{aligned} K_1 &= n(n-1)\int_0^1 \int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}} \left[1-c_{(2)}\right]\left[c_{(2)}-c_{(1)}\right] f\left[c_{(1)}\right] f\left[c_{(2)}\right]\left[1-F\left(c_{(2)}\right)\right]^{n-2} dc_{(2)}dc_{(1)} \\ &= n(n-1)x^2\int_0^1 \int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}} \left[1-c_{(2)}\right]\left[c_{(2)}-c_{(1)}\right]\left[1-c_{(2)}\right]^{x-1+(n-2)x}\left[1-c_{(1)}\right]^{x-1} dc_{(2)}dc_{(1)} \\ &= n(n-1)x^2\overbrace{\int_0^1 \left[1-c_{(1)}\right]^{x-1}\left[\int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}} c_{(2)}\left[1-c_{(2)}\right]^{(n-1)x} dc_{(2)}\right] dc_{(1)}}^{=K_{11}} \\ &\quad -n(n-1)x^2\underbrace{\int_0^1 c_{(1)}\left[1-c_{(1)}\right]^{x-1}\left[\int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}} \left[1-c_{(2)}\right]^{(n-1)x} dc_{(2)}\right] dc_{(1)}}_{=K_{12}}. \end{aligned}$$

Rewrite the K_{11} term, using Lemma 4 (the first equality) and Lemmas 1 and 2 (the second equality):

$$\begin{aligned}
K_{11} &= \int_0^1 \left[1-c_{(1)}\right]^{x-1} \left[\frac{c_{(1)}\left(1-c_{(1)}\right)^{1+(n-1)x}\left[1-\left(\frac{1}{2}\right)^{2+(n-1)x}\right]}{2+(n-1)x} \right. \\
&\quad \left. + \frac{\left(1-c_{(1)}\right)^{1+(n-1)x}\left[1-\left(\frac{1}{2}\right)^{2+(n-1)x}[3+(n-1)x]\right]}{[1+(n-1)x][2+(n-1)x]} \right] dc_{(1)} \\
&= \frac{1-\left(\frac{1}{2}\right)^{2+(n-1)x}}{(1+nx)(2+nx)[2+(n-1)x]} + \frac{1-\left(\frac{1}{2}\right)^{2+(n-1)x}[3+(n-1)x]}{(1+nx)[1+(n-1)x][2+(n-1)x]} \\
&= \frac{[1+(n-1)x]\left[1-\left(\frac{1}{2}\right)^{2+(n-1)x}\right]+(2+nx)\left[1-\left(\frac{1}{2}\right)^{2+(n-1)x}[3+(n-1)x]\right]}{(1+nx)(2+nx)[1+(n-1)x][2+(n-1)x]} \\
&= \frac{[1+(n-1)x]+(2+nx)-\{[1+(n-1)x]+(2+nx)[3+(n-1)x]\}\left(\frac{1}{2}\right)^{2+(n-1)x}}{(1+nx)(2+nx)[1+(n-1)x][2+(n-1)x]}.
\end{aligned}$$

Rewrite the term K_{12}, using Lemma 5 (the first equality) and Lemma 2 (the second equality):

$$\begin{aligned}
K_{12} &= \int_0^1 c_{(1)}\left[1-c_{(1)}\right]^{x-1}\left[\frac{\left(1-c_{(1)}\right)^{1+(n-1)x}\left[1-\left(\frac{1}{2}\right)^{1+(n-1)x}\right]}{1+(n-1)x}\right] dc_{(1)} \\
&= \frac{1-\left(\frac{1}{2}\right)^{1+(n-1)x}}{(1+nx)(2+nx)[1+(n-1)x]}.
\end{aligned}$$

Now rewrite the second term, K_2:

$$\begin{aligned}
K_2 &= \int_0^1 \int_{\frac{1+c_{(1)}}{2}}^1 \left[1-\frac{1+c_{(1)}}{2}\right]\left[\frac{1+c_{(1)}}{2}-c_{(1)}\right] g_{1,2}\left[c_{(1)},c_{(2)}\right] dc_{(2)}dc_{(1)} \\
&= \frac{n(n-1)}{4}\int_0^1 \int_{\frac{1+c_{(1)}}{2}}^1 \left[1-c_{(1)}\right]^2 f\left[c_{(1)}\right] f\left[c_{(2)}\right]\left[1-F\left(c_{(2)}\right)\right]^{n-2} dc_{(2)}dc_{(1)} \\
&= \frac{n(n-1)x^2}{4}\int_0^1 \left[1-c_{(1)}\right]^2\left[1-c_{(1)}\right]^{x-1}\left[\int_{\frac{1+c_{(1)}}{2}}^1 \left[1-c_{(2)}\right]^{x-1+(n-2)x} dc_{(2)}\right] dc_{(1)} \\
&= \frac{n(n-1)x^2}{4(n-1)x}\int_0^1 \left[1-c_{(1)}\right]^{x+1}\left[\left(\frac{1-c_{(1)}}{2}\right)^{(n-1)x}\right] dc_{(1)} \\
&= \frac{nx\left(\frac{1}{2}\right)^{(n-1)x}}{4}\int_0^1 \left[1-c_{(1)}\right]^{nx+1} dc_{(1)} = \frac{nx\left(\frac{1}{2}\right)^{(n-1)x+2}}{2+nx},
\end{aligned}$$

where the second equality uses (3), the fourth equality uses Lemma 3, and the last equality uses Lemma 1.

Summing up we have

$$\begin{aligned}
E\left[\Pi^{CI}\right] &= K_1+K_2=n(n-1)x^2(K_{11}-K_{12})+K_2\\
&= n(n-1)x^2\times\\
&\quad\frac{[1+(n-1)x]+(2+nx)-\{[1+(n-1)x]+(2+nx)[3+(n-1)x]\}\left(\frac{1}{2}\right)^{2+(n-1)x}}{(1+nx)(2+nx)[1+(n-1)x][2+(n-1)x]}\\
&\quad -n(n-1)x^2\frac{1-\left(\frac{1}{2}\right)^{1+(n-1)x}}{(1+nx)(2+nx)[1+(n-1)x]}+\frac{nx\left(\frac{1}{2}\right)^{(n-1)x+2}}{2+nx}. \qquad (13)
\end{aligned}$$

The terms in (13) that do not involve $\left(\frac{1}{2}\right)^{(n-1)x+2}$ or $\left(\frac{1}{2}\right)^{(n-1)x+1}$ become:

$$\begin{aligned}
&\frac{n(n-1)x^2\{1+(n-1)x+(2+nx)-[(n-1)x+2]\}}{(nx+1)(nx+2)[(n-1)x+1][(n-1)x+2]}\\
=\ &\frac{n(n-1)x^2(1+nx)}{(nx+1)(nx+2)[(n-1)x+1][(n-1)x+2]}=\frac{n(n-1)x^2}{(nx+2)[(n-1)x+1][(n-1)x+2]}. \qquad (14)
\end{aligned}$$

The other terms in (13) except the last one become:

$$L_A=-\frac{n(n-1)x^2\left(\frac{1}{2}\right)^{(n-1)x+2}}{(nx+1)(nx+2)[(n-1)x+1][(n-1)x+2]}\times L_B$$

where

$$\begin{aligned}
L_B &= [1+(n-1)x]+(2+nx)[3+(n-1)x]-2[2+(n-1)x]\\
&= (n-1)x[1+(2+nx)-2]+1+3(2+nx)-4\\
&= (n-1)x(nx+1)+3(nx+1)=[(n-1)x+3](nx+1).
\end{aligned}$$

So

$$L_A=-\frac{n(n-1)x^2[(n-1)x+3]\left(\frac{1}{2}\right)^{(n-1)x+2}}{(nx+2)[(n-1)x+1][(n-1)x+2]}.$$

Adding L_A to the last term in (13):

$$\begin{aligned}
&\frac{nx\left(\frac{1}{2}\right)^{(n-1)x+2}}{nx+2}+L_A\\
=\ &\frac{nx\left(\frac{1}{2}\right)^{(n-1)x+2}}{2+nx}-\frac{n(n-1)x^2[(n-1)x+3]\left(\frac{1}{2}\right)^{(n-1)x+2}}{(nx+2)[(n-1)x+1][(n-1)x+2]}\\
=\ &\frac{nx\left(\frac{1}{2}\right)^{(n-1)x+2}\{[(n-1)x+1][(n-1)x+2]-(n-1)x[(n-1)x+3]\}}{(nx+2)[(n-1)x+1][(n-1)x+2]}\\
=\ &\frac{nx\left(\frac{1}{2}\right)^{(n-1)x+1}}{(nx+2)[(n-1)x+1][(n-1)x+2]}. \qquad (15)
\end{aligned}$$

Finally, adding (14) and (15) we get

$$\begin{aligned}
E\left[\Pi^{CI}\right] &= \frac{n(n-1)x^2}{(nx+2)[(n-1)x+1][(n-1)x+2]}+\frac{nx\left(\frac{1}{2}\right)^{(n-1)x+1}}{(nx+2)[(n-1)x+1][(n-1)x+2]}\\
&= \frac{nx\left[(n-1)x+\left(\frac{1}{2}\right)^{(n-1)x+1}\right]}{(nx+2)[(n-1)x+1][(n-1)x+2]},
\end{aligned}$$

which is identical to the expression in Table 1.

2.4 Expected consumer surplus

The expected consumer surplus can be calculated as

$$E\left[S^{CI}\right] = \overbrace{\int_0^1 \int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}} \frac{\left[1-c_{(2)}\right]^2}{2} g_{1,2}\left[c_{(1)}, c_{(2)}\right] dc_{(2)} dc_{(1)}}^{=M_1}$$
$$\underbrace{+\int_0^1 \int_{\frac{1+c_{(1)}}{2}}^1 \frac{1}{2}\left[1-\frac{1+c_{(1)}}{2}\right]^2 g_{1,2}\left[c_{(1)}, c_{(2)}\right] dc_{(2)} dc_{(1)}}_{=M_2}.$$

Rewrite the first term, M_1:

$$\begin{aligned}
& M_1 \\
= & \frac{n(n-1)}{2} \int_0^1 \int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}} \left[1-c_{(2)}\right]^2 f\left[c_{(1)}\right] f\left[c_{(2)}\right] \left[1-F\left(c_{(2)}\right)\right]^{n-2} dc_{(2)} dc_{(1)} \\
= & \frac{n(n-1)x^2}{2} \int_0^1 \left[1-c_{(1)}\right]^{x-1} \left[\int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}} \left[1-c_{(2)}\right]^2 \left[1-c_{(2)}\right]^{x-1} \left[1-c_{(2)}\right]^{x(n-2)} dc_{(2)}\right] dc_{(1)} \\
= & \frac{n(n-1)x^2}{2} \int_0^1 \left[1-c_{(1)}\right]^{x-1} \left[\int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}} \left[1-c_{(2)}\right]^{x(n-1)+1} dc_{(2)}\right] dc_{(1)} \\
= & \frac{n(n-1)x^2}{2} \int_0^1 \left[1-c_{(1)}\right]^{x-1} \left[\frac{\left(1-c_{(1)}\right)^{x(n-1)+2}\left[1-\left(\frac{1}{2}\right)^{x(n-1)+2}\right]}{x(n-1)+2}\right] dc_{(1)} \\
= & \frac{n(n-1)x^2\left[1-\left(\frac{1}{2}\right)^{x(n-1)+2}\right]}{2\left[x(n-1)+2\right]} \int_0^1 \left[1-c_{(1)}\right]^{nx+1} dc_{(1)} \\
= & \frac{n(n-1)x^2\left[1-\left(\frac{1}{2}\right)^{x(n-1)+2}\right]}{2(nx+2)\left[x(n-1)+2\right]},
\end{aligned} \tag{16}$$

where the first equality uses (3), the fourth equality uses Lemma 5, and the last equality uses Lemma 1.

Rewrite the second term, M_2:

$$\begin{aligned}
M_2 &= \frac{n(n-1)}{8}\int_0^1 \left[1-c_{(1)}\right]^2 f\left[c_{(1)}\right]\left[\int_{\frac{1+c_{(1)}}{2}}^1 f\left[c_{(2)}\right]\left[1-F\left(c_{(2)}\right)\right]^{n-2} dc_{(2)}\right] dc_{(1)} \\
&= \frac{n(n-1)x^2}{8}\int_0^1 \left[1-c_{(1)}\right]^{x+1}\left[\int_{\frac{1+c_{(1)}}{2}}^1 \left[1-c_{(2)}\right]^{x-1}\left[1-c_{(2)}\right]^{x(n-2)} dc_{(2)}\right] dc_{(1)} \\
&= \frac{n(n-1)x^2}{8}\int_0^1 \left[1-c_{(1)}\right]^{x+1}\left[\int_{\frac{1+c_{(1)}}{2}}^1 \left[1-c_{(2)}\right]^{x(n-1)-1} dc_{(2)}\right] dc_{(1)} \\
&= \frac{n(n-1)x^2}{8}\int_0^1 \left[1-c_{(1)}\right]^{x+1}\left[\frac{\left(\frac{1-c_{(1)}}{2}\right)^{x(n-1)}}{x(n-1)}\right] dc_{(1)} = \frac{nx\left(\frac{1}{2}\right)^{x(n-1)}}{8}\int_0^1 \left[1-c_{(1)}\right]^{xn+1} dc_{(1)} \\
&= \frac{nx\left(\frac{1}{2}\right)^{x(n-1)}}{8(nx+2)} = \frac{nx\left(\frac{1}{2}\right)^{x(n-1)+2}}{2(nx+2)}, \qquad (17)
\end{aligned}$$

where the first equality uses (3), the fourth equality uses Lemma 3, and the last equality uses Lemma 1.

Finally adding (16) and (17), we get

$$\begin{aligned}
E\left[S^{CI}\right] &= \frac{n(n-1)x^2\left[1-\left(\frac{1}{2}\right)^{x(n-1)+2}\right]}{2(nx+2)\left[x(n-1)+2\right]} + \frac{nx\left(\frac{1}{2}\right)^{x(n-1)+2}}{2(nx+2)} \\
&= \frac{nx\left\{(n-1)x + \left[x(n-1)+2-(n-1)x\right]\left(\frac{1}{2}\right)^{x(n-1)+2}\right\}}{2(nx+2)\left[x(n-1)+2\right]} \\
&= \frac{nx\left[(n-1)x + \left(\frac{1}{2}\right)^{x(n-1)+1}\right]}{2(nx+2)\left[x(n-1)+2\right]},
\end{aligned}$$

which is identical to the expression in Table 1.

2.5 Expected total surplus

The expected total surplus can be calculated as

$$\begin{aligned}
E\left[W^{CI}\right] &= E\left[S^{CI}\right] + E\left[\Pi^{CI}\right] \\
&= \frac{nx\left[(n-1)x + \left(\frac{1}{2}\right)^{x(n-1)+1}\right]}{2(nx+2)\left[x(n-1)+2\right]} + \frac{nx\left[(n-1)x + \left(\frac{1}{2}\right)^{(n-1)x+1}\right]}{(nx+2)\left[(n-1)x+1\right]\left[(n-1)x+2\right]} \\
&= \frac{nx\left[(n-1)x + \left(\frac{1}{2}\right)^{x(n-1)+1}\right]\left[(n-1)x+1\right]}{2(nx+2)\left[(n-1)x+1\right]\left[x(n-1)+2\right]} + \frac{2nx\left[(n-1)x + \left(\frac{1}{2}\right)^{(n-1)x+1}\right]}{2(nx+2)\left[(n-1)x+1\right]\left[(n-1)x+2\right]} \\
&= \frac{nx\left[(n-1)x+3\right]\left[(n-1)x + \left(\frac{1}{2}\right)^{x(n-1)+1}\right]}{2(nx+2)\left[(n-1)x+1\right]\left[x(n-1)+2\right]},
\end{aligned}$$

which is identical to the expression in Table 1.

References

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