Supplementary Material to
"Bertrand under Uncertainty: Private and Common Costs"

Johan N. M. Lagerlöf

University of Copenhagen

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Contents

1. Introduction

In this supplementary material, which is not meant to be published, I provide some proofs that were omitted from Lagerlöf (2016). In the next section I first state and prove some useful lemmas. After that I derive a cost expression stated in eq. (5) in Lagerlöf (2016) and I prove Proposition 2 and Result 1 of Lagerlöf (2016).

2. Preliminaries

Lemma S1. *For any $a > 0$ we have*

$$\int_0^1 (1-s)^a \, ds = \frac{1}{1+a}.$$

Proof: Straightforward and hence omitted.

Lemma S2. *For any $a > 0$ we have*

$$\int_0^1 s\,(1-s)^a \, ds = \frac{1}{(1+a)\,(2+a)}. \tag{1}$$

Proof: By differentiating the right-hand side, one can verify that the following holds:

$$\int s\,(1-s)^a \, ds = -\,(1-s)^{a+1}\,\frac{(a+1)\,s+1}{(1+a)\,(2+a)}. \tag{2}$$

Given (2), (1) follows immediately. □

Lemma S3. *For any $a > 0$ and any $b \in [0, 1]$ we have*

$$\int_{\frac{1+b}{2}}^1 (1-s)^a \, ds = \frac{\left(\frac{1-b}{2}\right)^{a+1}}{1+a}. \tag{3}$$

Proof:

$$\int_{\frac{1+b}{2}}^1 (1-s)^a \, ds = \frac{1}{1+a}\left[-\,(1-s)^{a+1}\right]_{\frac{1+b}{2}}^1 = \frac{\left(1-\frac{1+b}{2}\right)^{a+1}}{1+a},$$

which simplifies to the expression in (3). □

Lemma S4. *For any $a > 0$ and any $b \in [0, 1]$ we have*

$$\int_b^{\frac{1+b}{2}} s\,(1-s)^a \, ds \;=\; \frac{b\,(1-b)^{1+a}\left[1-\left(\frac{1}{2}\right)^{2+a}\right]}{2+a} + \frac{(1-b)^{1+a}\left[1-\left(\frac{1}{2}\right)^{2+a}(3+a)\right]}{(1+a)\,(2+a)}. \tag{4}$$

Proof: Using (2) we can write

$$\begin{aligned}
&\int_b^{\frac{1+b}{2}} s\,(1-s)^a \, ds \\
=\;& \left[-\,(1-s)^{a+1}\,\frac{(a+1)\,s+1}{(1+a)\,(2+a)}\right]_b^{\frac{1+b}{2}} \\
=\;& (1-b)^{a+1}\,\frac{(a+1)\,b+1}{(1+a)\,(2+a)} - \left(\frac{1-b}{2}\right)^{a+1}\frac{(a+1)\,\frac{1+b}{2}+1}{(1+a)\,(2+a)} \\
=\;& \frac{(1-b)^{a+1}}{(1+a)\,(2+a)}\left[(a+1)\,b+1-\left(\frac{1}{2}\right)^{a+2}\left[(a+1)\,(1+b)+2\right]\right],
\end{aligned}$$

which simplifies to (4). □

Lemma S5. *For any $a > 0$ and any $b \in [0, 1]$ we have*

$$\int_b^{\frac{1+b}{2}} (1-s)^a \, ds = \frac{(1-b)^{a+1}\left[1-\left(\frac{1}{2}\right)^{a+1}\right]}{1+a}. \tag{5}$$

Proof:

$$\begin{aligned}
\int_b^{\frac{1+b}{2}} (1-s)^a \, ds \;&=\; \left[-\frac{(1-s)^{a+1}}{1+a}\right]_b^{\frac{1+b}{2}} = \frac{(1-b)^{a+1}}{1+a} - \frac{\left(1-\frac{1+b}{2}\right)^{a+1}}{1+a} \\
&=\; \frac{(1-b)^{a+1}}{1+a} - \frac{(1-b)^{a+1}\left(\frac{1}{2}\right)^{a+1}}{1+a},
\end{aligned}$$

which simplifies to (5). □

3. Derivation of firm i's expected cost, stated in eq. (5) of Lagerlöf (2016)

Compute firm i's expected cost, conditional on being the cheapest firm. By using the definition of firm i's cost, stated in eq. (2) in Lagerlöf (2016), we have

$$\mathbb{E}\left[c_i \mid \text{firm } i \text{ wins}\right] = (1-\alpha)\, s_i + \frac{\alpha}{n-1}\sum \mathbb{E}\left[s_j \mid \text{firm } i \text{ wins}\right]. \tag{6}$$

In particular, we can calculate

$$\begin{aligned}
\mathbb{E}\left[s_j \mid \text{firm } i \text{ wins}\right] &= \frac{1}{1-F\left[\chi(p_i)\right]}\int_{\chi(p_i)}^{1} sf(s)\,ds = \frac{1}{\left[1-\chi(p_i)\right]^x}\int_{\chi(p_i)}^{1} sx(1-s)^{x-1}\,ds \\
&= \frac{1}{\left[1-\chi(p_i)\right]^x}\left[-s(1-s)^x \mid_{\chi(p_i)}^1 + \int_{\chi(p_i)}^{1}(1-s)^x\,ds\right] \\
&= \frac{1}{\left[1-\chi(p_i)\right]^x}\left[\chi(p_i)\left[1-\chi(p_i)\right]^x - \frac{(1-s)^{1+x}}{1+x}\mid_{\chi(p_i)}^1\right] \\
&= \frac{1}{\left[1-\chi(p_i)\right]^x}\left[\chi(p_i)\left[1-\chi(p_i)\right]^x + \frac{\left[1-\chi(p_i)\right]^{1+x}}{1+x}\right] \\
&= \chi(p_i) + \frac{1-\chi(p_i)}{1+x} = \frac{1+x\chi(p_i)}{1+x},
\end{aligned}$$

where the expression for the density $f(s)$ is obtained from (8). Substituting the derived expression for $\mathbb{E}\left[s_j \mid \text{firm } i \text{ wins}\right]$ back into (6), we obtain

$$\mathbb{E}[c_i \mid \text{firm } i \text{ wins}] = (1-\alpha)s_i + \alpha\left(\frac{1+x\chi(p_i)}{1+x}\right).$$

□

In Lagerlöf (2016) it was assumed that the firms' signals are independently drawn from the cumulative distribution function

$$F(s_i) = 1-(1-s_i)^x, \tag{7}$$

with support $[0,1]$ and with $x>0$ being a parameter; the associated density function is given by

$$f(s_i) = x(1-s_i)^{x-1}. \tag{8}$$

Therefore, the expected value of one draw s from the distribution in (7) is given by

$$\mathbb{E}[s] = \int_0^1 sx(1-s)^{x-1}\,ds = \frac{1}{1+x}. \tag{9}$$

The density function of the lowest among n draws, $s_{(1)}$, is given by[1]

$$g_1\left[s_{(1)}\right] = n\left[1-F\left[s_{(1)}\right]\right]^{n-1} f\left[s_{(1)}\right] = nx\left[1-s_{(1)}\right]^{nx-1}, \tag{10}$$

where $F(s_i)$ and $f(s_i)$ are given by (7) and (8). The expected value of $s_{(1)}$ is therefore given by

$$\begin{aligned}
\mathbb{E}\left[s_{(1)}\right] &= \int_0^1 s_{(1)}g_1\left[s_{(1)}\right]ds_{(1)} = \int_0^1 s_{(1)}n\left[1-F\left[s_{(1)}\right]\right]^{n-1}f\left[s_{(1)}\right]ds_{(1)} \\
&= -\left[1-F\left[s_{(1)}\right]\right]^n s_{(1)}\mid_0^1 + \int_0^1\left[1-F\left[s_{(1)}\right]\right]^n ds_{(1)} \\
&= \int_0^1\left[1-F\left[s_{(1)}\right]\right]^n ds_{(1)},
\end{aligned}$$

[1] In order to derive this expression for g_1, let G_1 be the cumulative distribution function associated with the density g_1. The probability that n independent draws all yield a cost equal to s or higher is given by $1-G_1(s) = [1-F(s)]^n$, so $G_1(s) = 1-[1-F(s)]^n$ and $g_1(s) = n[1-F(s)]^{n-1}f(s)$.

where the third equality is obtained by integrating by parts. Now, using the specific functional form that is assumed, we have

$$\mathbb{E}\left[s_{(1)}\right]=\int_{0}^{1}\left[1-s_{(1)}\right]^{nx}ds_{(1)}=\frac{1}{1+nx}. \tag{11}$$

The density function of the second lowest among n draws, $s_{(2)}$, is given by[2]

$$\begin{aligned}
g_{(2)}(s) &= \frac{n!}{(n-2)!}F(s)\left[1-F(s)\right]^{n-2}f(s)\\
&= n(n-1)\left[1-(1-s)^{x}\right](1-s)^{x(n-2)}x(1-s)^{x-1}\\
&= xn(n-1)\left[(1-s)^{x(n-1)-1}-(1-s)^{xn-1}\right].
\end{aligned}$$

The expected value of $s_{(2)}$ is therefore given by

$$\begin{aligned}
\mathbb{E}\left[s_{(2)}\right] &= \int_{0}^{1}s_{(2)}g_{(2)}ds_{(2)}\\
&= xn(n-1)\left[\int_{0}^{1}s(1-s)^{x(n-1)-1}ds-\int_{0}^{1}s(1-s)^{xn-1}ds\right]\\
&= xn(n-1)\left[\frac{1}{x(n-1)\left[x(n-1)+1\right]}-\frac{1}{xn(xn+1)}\right]\\
&= \frac{n}{x(n-1)+1}-\frac{n-1}{xn+1}\\
&= \frac{n(xn+1)-(n-1)\left[x(n-1)+1\right]}{\left[x(n-1)+1\right](xn+1)}\\
&= \frac{n(xn+1)-(n-1)\left[xn+1\right]+(n-1)x}{\left[x(n-1)+1\right](xn+1)}\\
&= \frac{\left[xn+1\right]+(n-1)x}{\left[x(n-1)+1\right](xn+1)}=\frac{(2n-1)x+1}{\left[x(n-1)+1\right](xn+1)}.
\end{aligned} \tag{12}$$

We can rewrite the intercept of the equilibrium price strategy, A, as follows,

$$\begin{aligned}
A &= \frac{1-\alpha}{1+\epsilon+(n-1)x}+\frac{\alpha}{1+x}=\frac{(1-\alpha)(1+x)+\alpha\left[1+\epsilon+(n-1)x\right]}{\left[1+\epsilon+(n-1)x\right](1+x)}\\
&= \frac{1+x+\alpha\left[1+\epsilon+(n-1)x-(1+x)\right]}{\left[1+\epsilon+(n-1)x\right](1+x)}=\frac{1+x+\alpha\left[\epsilon+(n-2)x\right]}{\left[1+\epsilon+(n-1)x\right](1+x)}.
\end{aligned} \tag{13}$$

4. Proof of Proposition 2

In the subsections below I will calculate expected price, consumer surplus, industry profits and total surplus—first under complete information and then under incomplete information. The resulting expressions are summarized in Table 1.

Proposition 2 assumes that $(\alpha,\epsilon)=(0,1)$, which in particular means that $s_{(j)}=c_{(j)}$. I will therefore in the remainder of this section work with a cost notation rather than a signal notation. Given the parameter values assumed in Proposition 2, the market price (i.e., the price charged by the firm with the lowest cost draw) equals

$$p^{II}=A+(1-A)c_{(1)},\qquad \text{with } A=\frac{1}{(n-1)x+2}. \tag{14}$$

[2] See, for example, Wolfstetter (1999, p. 344).

Table 1: Expected price, profits, consumer surplus and total surplus

	Incomplete information	Complete information
$\mathbb{E}[p]$	$\frac{2(nx+1)-x}{(nx+1)[(n-1)x+2]}$	$\frac{(2n-1)x+1-nx\left(\frac{1}{2}\right)^{(n-1)x+1}}{(nx+1)[(n-1)x+1]}$
$\mathbb{E}[\Pi]$	$\frac{nx[(n-1)x+1]}{(nx+2)[(n-1)x+2]^2}$	$\frac{nx\left[(n-1)x+\left(\frac{1}{2}\right)^{(n-1)x+1}\right]}{(nx+2)[(n-1)x+1][(n-1)x+2]}$
$\mathbb{E}[S]$	$\frac{nx[(n-1)x+1]^2}{2(nx+2)[(n-1)x+2]^2}$	$\frac{nx\left[(n-1)x+\left(\frac{1}{2}\right)^{x(n-1)+1}\right]}{2(nx+2)[x(n-1)+2]}$
$\mathbb{E}[W]$	$\frac{nx[(n-1)x+1][(n-1)x+3]}{2(nx+2)[(n-1)x+2]^2}$	$\frac{nx[(n-1)x+3]\left[(n-1)x+\left(\frac{1}{2}\right)^{x(n-1)+1}\right]}{2(nx+2)[(n-1)x+1][x(n-1)+2]}$

Also recall from Lagerlöf (2016) that the equilibrium market price under complete information is

$$p^{CI} = \min\left\{c_{(2)}, \frac{1+c_{(1)}}{2}\right\}$$

and that the joint probability density function of $c_{(1)}$ and $c_{(2)}$ is given by [see, for example, Gumbel (1958/2004, p. 53)]

$$g_{1,2}\left[c_{(1)}, c_{(2)}\right] = n(n-1) f\left[c_{(1)}\right] f\left[c_{(2)}\right]\left[1-F\left(c_{(2)}\right)\right]^{n-2} \tag{15}$$

if $c_{(1)} \leq c_{(2)}$ and 0 otherwise.

4.1. Calculation of expected price under complete information

The expected price is

$$\begin{aligned}\mathbb{E}\left[p^{CI}\right] &= \underbrace{\int_0^1 \int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}} c_{(2)} g_{1,2}\left[c_{(1)}, c_{(2)}\right] dc_{(2)} dc_{(1)}}_{=D_1} \\ &\quad \underbrace{+\int_0^1 \int_{\frac{1+c_{(1)}}{2}}^1 \frac{1+c_{(1)}}{2} g_{1,2}\left[c_{(1)}, c_{(2)}\right] dc_{(2)} dc_{(1)}.}_{=D_2}\end{aligned} \tag{16}$$

Using (15), we can write the first term of (16), D_1, as

$$D_1 = n(n-1)\int_0^1 f\left[c_{(1)}\right]\left[\int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}} c_{(2)} f\left[c_{(2)}\right]\left[1-F\left(c_{(2)}\right)\right]^{n-2} dc_{(2)}\right] dc_{(1)}. \tag{17}$$

The "inside integral" of (17) becomes

$$\begin{aligned}&\int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}} c_{(2)} f\left[c_{(2)}\right]\left[1-F\left(c_{(2)}\right)\right]^{n-2} dc_{(2)} \\ =\ & x\int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}} c_{(2)}\left(1-c_{(2)}\right)^{x-1+(n-2)x} dc_{(2)} \\ =\ & x\frac{c_{(1)}\left(1-c_{(1)}\right)^{(n-1)x}\left[1-\left(\frac{1}{2}\right)^{(n-1)x+1}\right]}{(n-1)x+1} + x\frac{\left(1-c_{(1)}\right)^{(n-1)x}\left[1-\left(\frac{1}{2}\right)^{(n-1)x+1}\left[(n-1)x+2\right]\right]}{(n-1)x\left[(n-1)x+1\right]},\end{aligned}$$

where the last equality uses Lemma S4. Plugging this expression back into (17) one has

$$\begin{aligned}
& D_1 \\
= \; & \frac{n(n-1)x^2\left[1-\left(\frac{1}{2}\right)^{(n-1)x+1}\right]}{(n-1)x+1}\int_0^1 c_{(1)}\left(1-c_{(1)}\right)^{nx-1} \\
& +\frac{n(n-1)x^2\left[1-\left(\frac{1}{2}\right)^{(n-1)x+1}[(n-1)x+2]\right]}{(n-1)x[(n-1)x+1]}\int_0^1\left(1-c_{(1)}\right)^{nx-1}dc_{(1)} \\
= \; & \frac{(n-1)x\left[1-\left(\frac{1}{2}\right)^{(n-1)x+1}\right]}{(nx+1)[(n-1)x+1]}+\frac{\left[1-\left(\frac{1}{2}\right)^{(n-1)x+1}[(n-1)x+2]\right]}{[(n-1)x+1]} \\
= \; & \frac{(n-1)x+(nx+1)}{(nx+1)[(n-1)x+1]} \\
& -\frac{\{(n-1)x+(nx+1)[(n-1)x+2]\}\left(\frac{1}{2}\right)^{(n-1)x+1}}{(nx+1)[(n-1)x+1]},
\end{aligned} \tag{18}$$

where the second equality uses Lemmas S1 and S2. Using (15), we can write the second term of (16), D_2, as

$$\begin{aligned}
D_2 &= n(n-1)\int_0^1\frac{1+c_{(1)}}{2}f\left[c_{(1)}\right]\left[\int_{\frac{1+c_{(1)}}{2}}^1 f\left[c_{(2)}\right]\left[1-F\left(c_{(2)}\right)\right]^{n-2}dc_{(2)}\right]dc_{(1)} \\
&= n(n-1)x^2\int_0^1\frac{1+c_{(1)}}{2}\left(1-c_{(1)}\right)^{x-1}\left[\int_{\frac{1+c_{(1)}}{2}}^1\left(1-c_{(2)}\right)^{x-1+x(n-2)}dc_{(2)}\right]dc_{(1)} \\
&= \frac{n(n-1)x^2}{2}\int_0^1\left(1+c_{(1)}\right)\left(1-c_{(1)}\right)^{x-1}\left[\frac{\left(\frac{1-c_{(1)}}{2}\right)^{x(n-1)}}{x(n-1)}\right]dc_{(1)} \\
&= nx\left(\frac{1}{2}\right)^{x(n-1)+1}\left[\int_0^1\left(1-c_{(1)}\right)^{xn-1}dc_{(1)}+\int_0^1 c_{(1)}\left(1-c_{(1)}\right)^{xn-1}dc_{(1)}\right] \\
&= nx\left(\frac{1}{2}\right)^{x(n-1)+1}\left[\frac{1}{nx}+\frac{1}{nx(nx+1)}\right]=\frac{(nx+2)\left(\frac{1}{2}\right)^{x(n-1)+1}}{nx+1},
\end{aligned} \tag{19}$$

where the third equality uses Lemma S3 and the last equality uses Lemmas S1 and S2.

So summing up, using (18) and (19), we have

$$\begin{aligned}
\mathbb{E}\left[p^{CI}\right] = \; & \overbrace{\frac{(n-1)x+(nx+1)}{(nx+1)[(n-1)x+1]}-\frac{\{(n-1)x+(nx+1)[(n-1)x+2]\}\left(\frac{1}{2}\right)^{(n-1)x+1}}{(nx+1)[(n-1)x+1]}}^{\text{from (18)}} \\
& +\overbrace{\frac{(nx+2)[(n-1)x+1]\left(\frac{1}{2}\right)^{x(n-1)+1}}{(nx+1)[(n-1)x+1]}}^{\text{from (19)}}.
\end{aligned}$$

The numerator terms multiplied by $\left(\frac{1}{2}\right)^{(n-1)x+1}$ reduce to the following:

$$\begin{aligned}
& (nx+2)[(n-1)x+1]-\{(n-1)x+(nx+1)[(n-1)x+2]\} \\
= \; & (nx+2)[(n-1)x+1]-(n-1)x-(nx+1)[(n-1)x+1]-(nx+1) \\
= \; & [(n-1)x+1]-(n-1)x-(nx+1)=-nx.
\end{aligned}$$

Therefore,

$$\mathbb{E}\left[p^{CI}\right]=\frac{(2n-1)x+1}{(nx+1)[(n-1)x+1]}-\frac{nx\left(\frac{1}{2}\right)^{(n-1)x+1}}{(nx+1)[(n-1)x+1]},$$

which is identical to the expression in Table 1.

4.2. Calculation of expected industry profits under complete information

The expected industry profits are

$$
\begin{aligned}
\mathbb{E}\left[\Pi^{CI}\right] &= \overbrace{\int_0^1 \int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}} \left[1-c_{(2)}\right]\left[c_{(2)}-c_{(1)}\right] g_{1,2}\left[c_{(1)},c_{(2)}\right] dc_{(2)}dc_{(1)}}^{=K_1} \\
&\quad \underbrace{+\int_0^1 \int_{\frac{1+c_{(1)}}{2}}^{1} \left[1-\frac{1+c_{(1)}}{2}\right]\left[\frac{1+c_{(1)}}{2}-c_{(1)}\right] g_{1,2}\left[c_{(1)},c_{(2)}\right] dc_{(2)}dc_{(1)}}_{=K_2}.
\end{aligned}
$$

Rewrite the first term, K_1, using (15):

$$
\begin{aligned}
K_1 &= n(n-1)\int_0^1 \int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}} \left[1-c_{(2)}\right]\left[c_{(2)}-c_{(1)}\right] f\left[c_{(1)}\right] f\left[c_{(2)}\right]\left[1-F\left(c_{(2)}\right)\right]^{n-2} dc_{(2)}dc_{(1)} \\
&= n(n-1)x^2 \int_0^1 \int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}} \left[1-c_{(2)}\right]\left[c_{(2)}-c_{(1)}\right]\left[1-c_{(2)}\right]^{x-1+(n-2)x}\left[1-c_{(1)}\right]^{x-1} dc_{(2)}dc_{(1)} \\
&= n(n-1)x^2 \overbrace{\int_0^1 \left[1-c_{(1)}\right]^{x-1}\left[\int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}} c_{(2)}\left[1-c_{(2)}\right]^{(n-1)x} dc_{(2)}\right] dc_{(1)}}^{=K_{11}} \\
&\quad -n(n-1)x^2 \underbrace{\int_0^1 c_{(1)}\left[1-c_{(1)}\right]^{x-1}\left[\int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}} \left[1-c_{(2)}\right]^{(n-1)x} dc_{(2)}\right] dc_{(1)}}_{=K_{12}}.
\end{aligned}
$$

Rewrite the K_{11} term, using Lemma S4 (the first equality) and Lemmas S1 and S2 (the second equality):

$$
\begin{aligned}
K_{11} &= \int_0^1 \left[1-c_{(1)}\right]^{x-1}\left[\frac{c_{(1)}\left(1-c_{(1)}\right)^{1+(n-1)x}\left[1-\left(\frac{1}{2}\right)^{2+(n-1)x}\right]}{2+(n-1)x}\right. \\
&\quad \left. +\frac{\left(1-c_{(1)}\right)^{1+(n-1)x}\left[1-\left(\frac{1}{2}\right)^{2+(n-1)x}\left[3+(n-1)x\right]\right]}{\left[1+(n-1)x\right]\left[2+(n-1)x\right]}\right] dc_{(1)} \\
&= \frac{1-\left(\frac{1}{2}\right)^{2+(n-1)x}}{(1+nx)(2+nx)\left[2+(n-1)x\right]}+\frac{1-\left(\frac{1}{2}\right)^{2+(n-1)x}\left[3+(n-1)x\right]}{(1+nx)\left[1+(n-1)x\right]\left[2+(n-1)x\right]} \\
&= \frac{\left[1+(n-1)x\right]\left[1-\left(\frac{1}{2}\right)^{2+(n-1)x}\right]+(2+nx)\left[1-\left(\frac{1}{2}\right)^{2+(n-1)x}\left[3+(n-1)x\right]\right]}{(1+nx)(2+nx)\left[1+(n-1)x\right]\left[2+(n-1)x\right]} \\
&= \frac{\left[1+(n-1)x\right]+(2+nx)-\left\{\left[1+(n-1)x\right]+(2+nx)\left[3+(n-1)x\right]\right\}\left(\frac{1}{2}\right)^{2+(n-1)x}}{(1+nx)(2+nx)\left[1+(n-1)x\right]\left[2+(n-1)x\right]}.
\end{aligned}
$$

Rewrite the term K_{12}, using Lemma S5 (the first equality) and Lemma S2 (the second equality):

$$
\begin{aligned}
K_{12} &= \int_0^1 c_{(1)}\left[1-c_{(1)}\right]^{x-1}\left[\frac{\left(1-c_{(1)}\right)^{1+(n-1)x}\left[1-\left(\frac{1}{2}\right)^{1+(n-1)x}\right]}{1+(n-1)x}\right] dc_{(1)} \\
&= \frac{1-\left(\frac{1}{2}\right)^{1+(n-1)x}}{(1+nx)(2+nx)\left[1+(n-1)x\right]}.
\end{aligned}
$$

Now rewrite the second term, K_2:

$$\begin{aligned} K_2 &= \int_0^1 \int_{\frac{1+c_{(1)}}{2}}^1 \left[1 - \frac{1+c_{(1)}}{2}\right] \left[\frac{1+c_{(1)}}{2} - c_{(1)}\right] g_{1,2}\left[c_{(1)}, c_{(2)}\right] dc_{(2)} dc_{(1)} \\ &= \frac{n(n-1)}{4} \int_0^1 \int_{\frac{1+c_{(1)}}{2}}^1 \left[1 - c_{(1)}\right]^2 f\left[c_{(1)}\right] f\left[c_{(2)}\right] \left[1 - F\left(c_{(2)}\right)\right]^{n-2} dc_{(2)} dc_{(1)} \\ &= \frac{n(n-1)x^2}{4} \int_0^1 \left[1 - c_{(1)}\right]^2 \left[1 - c_{(1)}\right]^{x-1} \left[\int_{\frac{1+c_{(1)}}{2}}^1 \left[1 - c_{(2)}\right]^{x-1+(n-2)x} dc_{(2)}\right] dc_{(1)} \\ &= \frac{n(n-1)x^2}{4(n-1)x} \int_0^1 \left[1 - c_{(1)}\right]^{x+1} \left[\left(\frac{1-c_{(1)}}{2}\right)^{(n-1)x}\right] dc_{(1)} \\ &= \frac{nx\left(\frac{1}{2}\right)^{(n-1)x}}{4} \int_0^1 \left[1 - c_{(1)}\right]^{nx+1} dc_{(1)} = \frac{nx\left(\frac{1}{2}\right)^{(n-1)x+2}}{2+nx}, \end{aligned}$$

where the second equality uses (15), the fourth equality uses Lemma S3, and the last equality uses Lemma S1.

Summing up we have

$$\begin{aligned} \mathbb{E}\left[\Pi^{CI}\right] &= K_1 + K_2 = n(n-1)x^2(K_{11} - K_{12}) + K_2 \\ &= n(n-1)x^2 \times \\ &\quad \frac{[1+(n-1)x] + (2+nx) - \{[1+(n-1)x] + (2+nx)[3+(n-1)x]\}\left(\frac{1}{2}\right)^{2+(n-1)x}}{(1+nx)(2+nx)[1+(n-1)x][2+(n-1)x]} \\ &\quad -n(n-1)x^2 \frac{1-\left(\frac{1}{2}\right)^{1+(n-1)x}}{(1+nx)(2+nx)[1+(n-1)x]} + \frac{nx\left(\frac{1}{2}\right)^{(n-1)x+2}}{2+nx}. \end{aligned} \tag{20}$$

The terms in (20) that do not involve $\left(\frac{1}{2}\right)^{(n-1)x+2}$ or $\left(\frac{1}{2}\right)^{(n-1)x+1}$ become:

$$\begin{aligned} & \frac{n(n-1)x^2\{1+(n-1)x+(2+nx)-[(n-1)x+2]\}}{(nx+1)(nx+2)[(n-1)x+1][(n-1)x+2]} \\ = & \frac{n(n-1)x^2(1+nx)}{(nx+1)(nx+2)[(n-1)x+1][(n-1)x+2]} = \frac{n(n-1)x^2}{(nx+2)[(n-1)x+1][(n-1)x+2]}. \end{aligned} \tag{21}$$

The other terms in (20) except the last one become:

$$L_A = -\frac{n(n-1)x^2\left(\frac{1}{2}\right)^{(n-1)x+2}}{(nx+1)(nx+2)[(n-1)x+1][(n-1)x+2]} \times L_B$$

where

$$\begin{aligned} L_B &= [1+(n-1)x] + (2+nx)[3+(n-1)x] - 2[2+(n-1)x] \\ &= (n-1)x[1+(2+nx)-2] + 1 + 3(2+nx) - 4 \\ &= (n-1)x(nx+1) + 3(nx+1) = [(n-1)x+3](nx+1). \end{aligned}$$

So

$$L_A = -\frac{n(n-1)x^2[(n-1)x+3]\left(\frac{1}{2}\right)^{(n-1)x+2}}{(nx+2)[(n-1)x+1][(n-1)x+2]}.$$

Adding L_A to the last term in (20):

$$\begin{aligned}
& \frac{nx\left(\frac{1}{2}\right)^{(n-1)x+2}}{nx+2}+L_A \\
= & \frac{nx\left(\frac{1}{2}\right)^{(n-1)x+2}}{2+nx}-\frac{n\left(n-1\right)x^2\left[\left(n-1\right)x+3\right]\left(\frac{1}{2}\right)^{(n-1)x+2}}{\left(nx+2\right)\left[\left(n-1\right)x+1\right]\left[\left(n-1\right)x+2\right]} \\
= & \frac{nx\left(\frac{1}{2}\right)^{(n-1)x+2}\left\{\left[\left(n-1\right)x+1\right]\left[\left(n-1\right)x+2\right]-\left(n-1\right)x\left[\left(n-1\right)x+3\right]\right\}}{\left(nx+2\right)\left[\left(n-1\right)x+1\right]\left[\left(n-1\right)x+2\right]} \\
= & \frac{nx\left(\frac{1}{2}\right)^{(n-1)x+1}}{\left(nx+2\right)\left[\left(n-1\right)x+1\right]\left[\left(n-1\right)x+2\right]}.
\end{aligned} \tag{22}$$

Finally, adding (21) and (22) we get

$$\begin{aligned}
\mathbb{E}\left[\Pi^{CI}\right] &= \frac{n\left(n-1\right)x^2}{\left(nx+2\right)\left[\left(n-1\right)x+1\right]\left[\left(n-1\right)x+2\right]}+\frac{nx\left(\frac{1}{2}\right)^{(n-1)x+1}}{\left(nx+2\right)\left[\left(n-1\right)x+1\right]\left[\left(n-1\right)x+2\right]} \\
&= \frac{nx\left[\left(n-1\right)x+\left(\frac{1}{2}\right)^{(n-1)x+1}\right]}{\left(nx+2\right)\left[\left(n-1\right)x+1\right]\left[\left(n-1\right)x+2\right]},
\end{aligned}$$

which is identical to the expression in Table 1.

4.3. Calculation of expected consumer surplus under complete information

The expected consumer surplus can be calculated as

$$\begin{aligned}
\mathbb{E}\left[S^{CI}\right] = & \overbrace{\int_0^1\int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}}\frac{\left[1-c_{(2)}\right]^2}{2}g_{1,2}\left[c_{(1)},c_{(2)}\right]dc_{(2)}dc_{(1)}}^{=M_1} \\
& \underbrace{+\int_0^1\int_{\frac{1+c_{(1)}}{2}}^1\frac{1}{2}\left[1-\frac{1+c_{(1)}}{2}\right]^2 g_{1,2}\left[c_{(1)},c_{(2)}\right]dc_{(2)}dc_{(1)}}_{=M_2}.
\end{aligned}$$

Rewrite the first term, M_1:

$$\begin{aligned}
& M_1 \\
= & \frac{n\left(n-1\right)}{2}\int_0^1\int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}}\left[1-c_{(2)}\right]^2 f\left[c_{(1)}\right]f\left[c_{(2)}\right]\left[1-F\left(c_{(2)}\right)\right]^{n-2}dc_{(2)}dc_{(1)} \\
= & \frac{n\left(n-1\right)x^2}{2}\int_0^1\left[1-c_{(1)}\right]^{x-1}\left[\int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}}\left[1-c_{(2)}\right]^2\left[1-c_{(2)}\right]^{x-1}\left[1-c_{(2)}\right]^{x(n-2)}dc_{(2)}\right]dc_{(1)} \\
= & \frac{n\left(n-1\right)x^2}{2}\int_0^1\left[1-c_{(1)}\right]^{x-1}\left[\int_{c_{(1)}}^{\frac{1+c_{(1)}}{2}}\left[1-c_{(2)}\right]^{x(n-1)+1}dc_{(2)}\right]dc_{(1)} \\
= & \frac{n\left(n-1\right)x^2}{2}\int_0^1\left[1-c_{(1)}\right]^{x-1}\left[\frac{\left(1-c_{(1)}\right)^{x(n-1)+2}\left[1-\left(\frac{1}{2}\right)^{x(n-1)+2}\right]}{x\left(n-1\right)+2}\right]dc_{(1)} \\
= & \frac{n\left(n-1\right)x^2\left[1-\left(\frac{1}{2}\right)^{x(n-1)+2}\right]}{2\left[x\left(n-1\right)+2\right]}\int_0^1\left[1-c_{(1)}\right]^{nx+1}dc_{(1)} \\
= & \frac{n\left(n-1\right)x^2\left[1-\left(\frac{1}{2}\right)^{x(n-1)+2}\right]}{2\left(nx+2\right)\left[x\left(n-1\right)+2\right]},
\end{aligned} \tag{23}$$

where the first equality uses (15), the fourth equality uses Lemma S5, and the last equality uses Lemma S1.

Rewrite the second term, M_2:

$$\begin{aligned}
M_2 &= \frac{n(n-1)}{8}\int_0^1 \left[1-c_{(1)}\right]^2 f\left[c_{(1)}\right]\left[\int_{\frac{1+c_{(1)}}{2}}^1 f\left[c_{(2)}\right]\left[1-F\left(c_{(2)}\right)\right]^{n-2} dc_{(2)}\right] dc_{(1)} \\
&= \frac{n(n-1)x^2}{8}\int_0^1 \left[1-c_{(1)}\right]^{x+1}\left[\int_{\frac{1+c_{(1)}}{2}}^1 \left[1-c_{(2)}\right]^{x-1}\left[1-c_{(2)}\right]^{x(n-2)} dc_{(2)}\right] dc_{(1)} \\
&= \frac{n(n-1)x^2}{8}\int_0^1 \left[1-c_{(1)}\right]^{x+1}\left[\int_{\frac{1+c_{(1)}}{2}}^1 \left[1-c_{(2)}\right]^{x(n-1)-1} dc_{(2)}\right] dc_{(1)} \\
&= \frac{n(n-1)x^2}{8}\int_0^1 \left[1-c_{(1)}\right]^{x+1}\left[\frac{\left(\frac{1-c_{(1)}}{2}\right)^{x(n-1)}}{x(n-1)}\right] dc_{(1)} = \frac{nx\left(\frac{1}{2}\right)^{x(n-1)}}{8}\int_0^1 \left[1-c_{(1)}\right]^{xn+1} dc_{(1)} \\
&= \frac{nx\left(\frac{1}{2}\right)^{x(n-1)}}{8(nx+2)} = \frac{nx\left(\frac{1}{2}\right)^{x(n-1)+2}}{2(nx+2)},
\end{aligned} \tag{24}$$

where the first equality uses (15), the fourth equality uses Lemma S3, and the last equality uses Lemma S1.

Finally adding (23) and (24), we get

$$\begin{aligned}
\mathbb{E}\left[S^{CI}\right] &= \frac{n(n-1)x^2\left[1-\left(\frac{1}{2}\right)^{x(n-1)+2}\right]}{2(nx+2)\left[x(n-1)+2\right]} + \frac{nx\left(\frac{1}{2}\right)^{x(n-1)+2}}{2(nx+2)} \\
&= \frac{nx\left\{(n-1)x+\left[x(n-1)+2-(n-1)x\right]\left(\frac{1}{2}\right)^{x(n-1)+2}\right\}}{2(nx+2)\left[x(n-1)+2\right]} \\
&= \frac{nx\left[(n-1)x+\left(\frac{1}{2}\right)^{x(n-1)+1}\right]}{2(nx+2)\left[x(n-1)+2\right]},
\end{aligned}$$

which is identical to the expression in Table 1.

4.4. Calculation of expected total surplus under complete information

The expected total surplus can be calculated as

$$\begin{aligned}
\mathbb{E}\left[W^{CI}\right] &= \mathbb{E}\left[S^{CI}\right] + \mathbb{E}\left[\Pi^{CI}\right] \\
&= \frac{nx\left[(n-1)x+\left(\frac{1}{2}\right)^{x(n-1)+1}\right]}{2(nx+2)\left[x(n-1)+2\right]} + \frac{nx\left[(n-1)x+\left(\frac{1}{2}\right)^{(n-1)x+1}\right]}{(nx+2)\left[(n-1)x+1\right]\left[(n-1)x+2\right]} \\
&= \frac{nx\left[(n-1)x+\left(\frac{1}{2}\right)^{x(n-1)+1}\right]\left[(n-1)x+1\right]}{2(nx+2)\left[(n-1)x+1\right]\left[x(n-1)+2\right]} + \frac{2nx\left[(n-1)x+\left(\frac{1}{2}\right)^{(n-1)x+1}\right]}{2(nx+2)\left[(n-1)x+1\right]\left[(n-1)x+2\right]} \\
&= \frac{nx\left[(n-1)x+3\right]\left[(n-1)x+\left(\frac{1}{2}\right)^{x(n-1)+1}\right]}{2(nx+2)\left[(n-1)x+1\right]\left[x(n-1)+2\right]},
\end{aligned}$$

which is identical to the expression in Table 1.

4.5. Calculation of expected price under incomplete information

Using (11) and (14), we have that the expected value of the market price equals

$$\begin{aligned}
\mathbb{E}\left[p^{II}\right] &= \frac{1+\left[(n-1)x+1\right]E\left[c_{(1)}\right]}{(n-1)x+2} = \frac{(1+nx)+(n-1)x+1}{\left[(n-1)x+2\right](1+nx)} \\
&= \frac{2+(2n-1)x}{\left[(n-1)x+2\right](1+xn)},
\end{aligned}$$

which is identical to the expression in Table 1.

4.6. Calculation of expected industry profits under incomplete information

Expected industry profits under incomplete information equal

$$\begin{aligned}
\mathbb{E}\left[\Pi^{II}\right] &= \int_0^1 \left[p^{II} - c_{(1)}\right]\left(1 - p^{II}\right) g_1\left[c_{(1)}\right] dc_{(1)} \\
&= \int_0^1 A\left[1 - c_{(1)}\right](1 - A)\left[1 - c_{(1)}\right] g_1\left[c_{(1)}\right] dc_{(1)} \\
&= A(1 - A)\, nx \int_0^1 \left[1 - c_{(1)}\right]^{nx+1} dc_{(1)} = \frac{A(1 - A)\, nx}{nx + 2} \\
&= \frac{nx\left[(n-1)x + 1\right]}{(nx + 2)\left[(n-1)x + 2\right]^2},
\end{aligned}$$

where the second equality uses $p^{II} - c_{(1)} = A\left[1 - c_{(1)}\right]$ and $1 - p^{II} = (1 - A)\left[1 - c_{(1)}\right]$, the third one uses (10), and the last equality uses (14). The last line is identical to the expression in Table 1.

4.7. Calculation of expected consumer surplus under incomplete information

Expected consumer surplus under incomplete information equals

$$\begin{aligned}
\mathbb{E}\left[S^{II}\right] &= \int_0^1 \frac{\left(1 - p^{II}\right)^2}{2} g_1\left[c_{(1)}\right] dc_{(1)} \\
&= \int_0^1 \frac{(1 - A)^2\left[1 - c_{(1)}\right]^2}{2} g_1\left[c_{(1)}\right] dc_{(1)} \\
&= \frac{nx(1 - A)^2}{2} \int_0^1 \left[1 - c_{(1)}\right]^{nx+1} dc_{(1)} \\
&= \frac{nx(1 - A)^2}{2(nx + 2)} = \left[\frac{(n-1)x + 1}{(n-1)x + 2}\right]^2 \frac{nx}{2(nx + 2)},
\end{aligned}$$

where the second equality uses $1 - p^{II} = (1 - A)\left[1 - c_{(1)}\right]$, the third one uses (10), and the last equality uses (14). The last line is identical to the expression in Table 1.

4.8. Calculation of expected total surplus under incomplete information

Finally, using the above results, we have that expected total surplus is

$$\begin{aligned}
\mathbb{E}\left[W^{II}\right] &= \mathbb{E}\left[S^{II}\right] + \mathbb{E}\left[\Pi^{II}\right] = \\
&\left[\frac{(n-1)x + 1}{(n-1)x + 2}\right]^2 \frac{nx}{2(nx + 2)} + \frac{nx\left[(n-1)x + 1\right]}{(nx + 2)\left[(n-1)x + 2\right]^2} \\
&= \frac{nx\left[(n-1)x + 1\right]^2 + 2nx\left[(n-1)x + 1\right]}{2(nx + 2)\left[(n-1)x + 2\right]^2} \\
&= \frac{nx\left[(n-1)x + 1\right]\left[(n-1)x + 1 + 2\right]}{2(nx + 2)\left[(n-1)x + 2\right]^2} \\
&= \frac{nx\left[(n-1)x + 1\right]\left[(n-1)x + 3\right]}{2(nx + 2)\left[(n-1)x + 2\right]^2},
\end{aligned}$$

which is identical to the expression in Table 1.

4.9. Comparisons

Here I compare the expressions in the left column of Table 1 with the ones in the right column. First, however, I state and prove a lemma that will be used in the later derivations.

Lemma S6. *We have* $(n-1)\,x+2<2^{(n-1)x+1}$ *for all* $x>0$ *and* $n>1$.

Proof. We can write

$$(n-1)\,x+2<2^{(n-1)x+1} \Leftrightarrow \ln(y+2)<(y+1)\ln(2) \Leftrightarrow \xi(y)>0,$$

where

$$y \stackrel{\text{def}}{=} (n-1)\,x \qquad \text{and} \qquad \xi(y) \stackrel{\text{def}}{=} (y+1)\ln(2)-\ln(y+2)\,.$$

The result now follows from the facts that $\xi(0)=0$ and

$$\xi'(y)=\ln(2)-\frac{1}{y+2}>0$$

(where the last inequality holds because $\ln(2)>0.5$ and $y>0$). □

Consider the comparison of expected prices. Using the expressions in Table 1 we have

$$\mathbb{E}\left[p^{II}\right]<\mathbb{E}\left[p^{CI}\right] \Leftrightarrow \frac{2\,(nx+1)-x}{(nx+1)\,[(n-1)\,x+2]}<\frac{(2n-1)\,x+1-nx\left(\frac{1}{2}\right)^{(n-1)x+1}}{(nx+1)\,[(n-1)\,x+1]} \Leftrightarrow$$

$$[2\,(nx+1)-x]\,[(n-1)\,x+1]<\left[(2n-1)\,x+1-nx\left(\frac{1}{2}\right)^{(n-1)x+1}\right][(n-1)\,x+2] \Leftrightarrow$$

$$\begin{aligned}
& [(n-1)\,x+2]\,nx\left(\frac{1}{2}\right)^{(n-1)x+1} \\
< \;& [(2n-1)\,x+1]\,[(n-1)\,x+2]-[2\,(nx+1)-x]\,[(n-1)\,x+1] \\
= \;& (n-1)\,x\,\{[(2n-1)\,x+1]-[2\,(nx+1)-x]\}+2\,[(2n-1)\,x+1]-[2\,(nx+1)-x] \\
= \;& -(n-1)\,x+2nx-x=nx \Leftrightarrow
\end{aligned}$$

$$[(n-1)\,x+2]\left(\frac{1}{2}\right)^{(n-1)x+1}<1 \Leftrightarrow (n-1)\,x+2<2^{(n-1)x+1},$$

which we know holds for all $x>0$ and $n>1$ (see Lemma S6).

Next consider the comparison of expected industry profits in the two models. Using the expressions in Table 1 we have

$$\mathbb{E}\left[\Pi^{CI}\right]<\mathbb{E}\left[\Pi^{II}\right] \Leftrightarrow \frac{nx\left[(n-1)\,x+\left(\frac{1}{2}\right)^{(n-1)x+1}\right]}{(nx+2)\,[(n-1)\,x+1]\,[(n-1)\,x+2]}<\frac{nx\,[(n-1)\,x+1]}{(nx+2)\,[(n-1)\,x+2]^2} \Leftrightarrow$$

$$[(n-1)\,x+2]\left[(n-1)\,x+\left(\frac{1}{2}\right)^{(n-1)x+1}\right]<[(n-1)\,x+1]^2 \Leftrightarrow$$

$$[(n-1)\,x+2]\left(\frac{1}{2}\right)^{(n-1)x+1}<[(n-1)\,x+1]^2-[(n-1)\,x+2]\,(n-1)\,x=1 \Leftrightarrow$$

$$(n-1)\,x+2<2^{(n-1)x+1},$$

which we know holds for all $x>0$ and $n>1$ (see Lemma S6).

Now consider the comparison of expected consumer surplus in the two models. Using the expressions in Table 1 we have

$$\mathbb{E}\left[S^{CI}\right] < \mathbb{E}\left[S^{II}\right] \Leftrightarrow \frac{nx\left[(n-1)\,x + \left(\frac{1}{2}\right)^{x(n-1)+1}\right]}{2\left[x\,(n-1)+2\right](xn+2)} < \left[\frac{(n-1)\,x+1}{(n-1)\,x+2}\right]^2 \frac{nx}{2\,(nx+2)}.$$

By simplifying this inequality one can verify that it is equivalent to the inequality $\mathbb{E}\left[\Pi^{CI}\right] < \mathbb{E}\left[\Pi^{II}\right]$ above, which we saw holds for all $x > 0$ and $n > 1$.

Finally, the fact that $\mathbb{E}\left[W^{CI}\right] < \mathbb{E}\left[W^{II}\right]$ follows immediately from the results that $\mathbb{E}\left[S^{CI}\right] < \mathbb{E}\left[S^{II}\right]$ and $\mathbb{E}\left[\Pi^{CI}\right] < \mathbb{E}\left[\Pi^{II}\right]$.

5. Proof of Result 1

The result assumes that $\alpha = \frac{1}{2}$, which in particular means that $p^{CI} = c_{(2)}$ (i.e., there are no drastic innovations). Also note that, under our assumptions, the joint density of $s_{(1)}$ and $s_{(2)}$ is given by

$$g_{1,2}\left[s_{(1)}, s_{(2)}\right] = n\,(n-1)\,f\left(s_{(1)}\right) f\left(s_{(2)}\right)\left[1 - F\left(s_{(2)}\right)\right]^{n-2} = 2.$$

Similarly, the density of $s_{(1)}$ can be written as

$$g_1\left[s_{(1)}\right] = 2\left(1 - s_{(1)}\right).$$

Furthermore, under our assumptions, the intercept of the equilibrium strategy under incomplete information, A, can be written as (the parameter α will be set equal to $\frac{1}{2}$ later)

$$A = \frac{2 + \epsilon\alpha}{2\,(2+\epsilon)}.$$

5.1. Calculation of expected profits under complete and incomplete information

Given that $\alpha = \frac{1}{2}$, the two firms have the same cost. Hence, by standard arguments, under complete information each firms zero profits: $\mathbb{E}\left[\Pi^{CI}\right] = 0$.

Next compute industry profits under incomplete information, $\mathbb{E}\left[\Pi^{II}\right]$. We can write

$$\begin{aligned}
\mathbb{E}\left[\Pi^{II}\right] &= \int_0^1 \int_{s_{(1)}}^1 \left[1 - A - (1-A)\,s_{(1)}\right]^{\epsilon} \left[A + (1-A)\,s_{(1)} - c_{(1)}\right] g_{1,2}\left[s_{(1)}, s_{(2)}\right] ds_{(2)} ds_{(1)} \\
&= 2\int_0^1 \left[(1-A)\left(1 - s_{(1)}\right)\right]^{\epsilon} \int_{s_{(1)}}^1 \left[A + (1-A)\,s_{(1)} - (1-\alpha)\,s_{(1)} - \alpha s_{(2)}\right] ds_{(2)} ds_{(1)} \\
&= 2\int_0^1 \left[(1-A)\left(1 - s_{(1)}\right)\right]^{\epsilon} \int_{s_{(1)}}^1 \left[A + (\alpha - A)\,s_{(1)} - \alpha s_{(2)}\right] ds_{(2)} ds_{(1)} \\
&= 2\int_0^1 \left[(1-A)\left(1 - s_{(1)}\right)\right]^{\epsilon} \left[\left[A + (\alpha - A)\,s_{(1)}\right] \int_{s_{(1)}}^1 ds_{(2)} - \alpha \int_{s_{(1)}}^1 s_{(2)} ds_{(2)}\right] ds_{(1)} \\
&= 2\int_0^1 \left[(1-A)\left(1 - s_{(1)}\right)\right]^{\epsilon} \left[\left[A + (\alpha - A)\,s_{(1)}\right]\left[1 - s_{(1)}\right] - \alpha \frac{1 - s_{(1)}^2}{2}\right] ds_{(1)} \\
&= (1-A)^{\epsilon} \int_0^1 \left[\left(1 - s_{(1)}\right)\right]^{1+\epsilon} \left[2\left[A + (\alpha - A)\,s_{(1)}\right] - \alpha\left(1 + s_{(1)}\right)\right] ds_{(1)} \\
&= (1-A)^{\epsilon} \int_0^1 \left[\left(1 - s_{(1)}\right)\right]^{1+\epsilon} \left[2A - \alpha + (\alpha - 2A)\,s_{(1)}\right] ds_{(1)} \\
&= (1-A)^{\epsilon}\,(2A - \alpha) \int_0^1 \left[\left(1 - s_{(1)}\right)\right]^{2+\epsilon} ds_{(1)}.
\end{aligned}$$

We have

$$2A-\alpha=2\frac{2+\epsilon\alpha}{2(2+\epsilon)}-\alpha=\frac{2+\epsilon\alpha-(2+\epsilon)\alpha}{2+\epsilon}=\frac{2(1-\alpha)}{2+\epsilon}$$

and

$$1-A=1-\frac{2+\epsilon\alpha}{2(2+\epsilon)}=\frac{2(2+\epsilon)-(2+\epsilon\alpha)}{2(2+\epsilon)}=\frac{2+\epsilon(2-\alpha)}{2(2+\epsilon)}.$$

Hence

$$\begin{aligned}\mathbb{E}\left[\Pi^{II}\right] &= (1-A)^{\epsilon}(2A-\alpha)\int_0^1\left[\left(1-s_{(1)}\right)\right]^{2+\epsilon}ds_{(1)}\\ &= \frac{2(1-\alpha)}{2+\epsilon}\left[\frac{2+\epsilon(2-\alpha)}{2(2+\epsilon)}\right]^{\epsilon}\int_0^1\left[\left(1-s_{(1)}\right)\right]^{2+\epsilon}ds_{(1)}=\frac{(1-\alpha)\left[2+\epsilon(2-\alpha)\right]^{\epsilon}}{2^{\epsilon-1}(2+\epsilon)^{1+\epsilon}(3+\epsilon)}.\end{aligned}$$

Setting $\alpha=\frac{1}{2}$, this simplifies to

$$\mathbb{E}\left[\Pi^{II}\right]=\frac{\left[2+\epsilon\left(\frac{3}{2}\right)\right]^{\epsilon}}{2^{\epsilon}(2+\epsilon)^{1+\epsilon}(3+\epsilon)}=\frac{(4+3\epsilon)^{\epsilon}}{2^{2\epsilon}(2+\epsilon)^{1+\epsilon}(3+\epsilon)}.$$

5.2. Calculation of expected consumer surplus under complete and incomplete information

First calculate expected consumer surplus under complete information, $E\left[S^{CI}\right]$. We can write

$$\begin{aligned}\mathbb{E}\left[S^{CI}\right] &= \frac{2}{1+\epsilon}\int_0^1\int_{s_{(1)}}^1\left[1-c_{(2)}\right]^{1+\epsilon}ds_{(2)}ds_{(1)}=\frac{2}{1+\epsilon}\int_0^1\int_{s_{(1)}}^1\left[1-(1-\alpha)s_{(2)}-\alpha s_{(1)}\right]^{1+\epsilon}ds_{(2)}ds_{(1)}\\ &= \frac{-2}{(1+\epsilon)(2+\epsilon)(1-\alpha)}\int_0^1\left[\left[1-(1-\alpha)s_{(2)}-\alpha s_{(1)}\right]^{2+\epsilon}\right]_{s_{(1)}}^1 ds_{(1)}\\ &= \frac{2}{(1+\epsilon)(2+\epsilon)(1-\alpha)}\int_0^1\left[\left[1-(1-\alpha)s_{(1)}-\alpha s_{(1)}\right]^{2+\epsilon}-\left[1-(1-\alpha)-\alpha s_{(1)}\right]^{2+\epsilon}\right]ds_{(1)}\\ &= 2\int_0^1\frac{\left(1-s_{(1)}\right)^{2+\epsilon}-\left[\alpha\left(1-s_{(1)}\right)\right]^{2+\epsilon}}{(1+\epsilon)(2+\epsilon)(1-\alpha)}ds_{(1)}=\frac{2\left(1-\alpha^{2+\epsilon}\right)}{(1+\epsilon)(2+\epsilon)(1-\alpha)}\int_0^1\left(1-s_{(1)}\right)^{2+\epsilon}ds_{(1)}\\ &= \frac{2\left(1-\alpha^{2+\epsilon}\right)}{(1+\epsilon)(2+\epsilon)(3+\epsilon)(1-\alpha)}\end{aligned}$$

or, with $\alpha=\frac{1}{2}$,

$$\mathbb{E}\left[S^{CI}\right]=\frac{4\left(1-\left(\frac{1}{2}\right)^{2+\epsilon}\right)}{(1+\epsilon)(2+\epsilon)(3+\epsilon)}=\frac{4-\frac{1}{2^{\epsilon}}}{(1+\epsilon)(2+\epsilon)(3+\epsilon)}.$$

Next compute expected consumer surplus under incomplete information, $E\left[S^{II}\right]$. We can write

$$\begin{aligned}\mathbb{E}\left[S^{II}\right] &= \frac{1}{1+\epsilon}\int_0^1\left[1-A-(1-A)s_{(1)}\right]^{1+\epsilon}g_1\left[s_{(1)}\right]ds_{(1)}\\ &= \frac{1}{1+\epsilon}\int_0^1\left[1-A-(1-A)s_{(1)}\right]^{1+\epsilon}\left[2\left(1-s_{(1)}\right)\right]ds_{(1)}\\ &= \frac{2(1-A)^{1+\epsilon}}{1+\epsilon}\int_0^1\left(1-s_{(1)}\right)^{2+\epsilon}ds_{(1)}=\frac{2(1-A)^{1+\epsilon}}{(1+\epsilon)(3+\epsilon)}=\frac{2\left(1-\frac{2+\epsilon\alpha}{2(2+\epsilon)}\right)^{1+\epsilon}}{(1+\epsilon)(3+\epsilon)}\\ &= \frac{\left[2+\epsilon(2-\alpha)\right]^{1+\epsilon}}{2^{\epsilon}(1+\epsilon)(3+\epsilon)(2+\epsilon)^{1+\epsilon}}\end{aligned}$$

or, with $\alpha=\frac{1}{2}$,

$$\mathbb{E}\left[S^{II}\right]=\frac{\left[2+\epsilon\left(\frac{3}{2}\right)\right]^{1+\epsilon}}{2^{\epsilon}(1+\epsilon)(3+\epsilon)(2+\epsilon)^{1+\epsilon}}=\frac{(4+3\epsilon)^{1+\epsilon}}{2^{1+2\epsilon}(1+\epsilon)(3+\epsilon)(2+\epsilon)^{1+\epsilon}}.$$

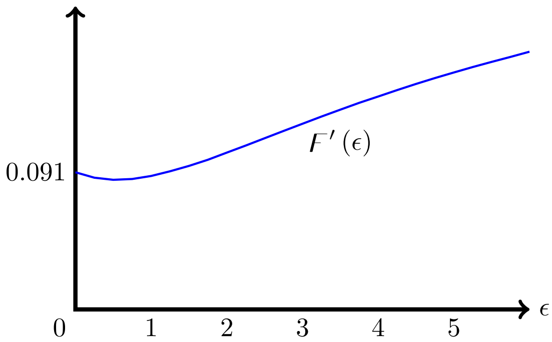

Figure 1: The graph of $F'(\epsilon)$, which suggests that $F'(\epsilon) > 0$ for all $\epsilon \geq 0$.

5.3. Comparison of expected total surplus

By adding up the results obtained above, we can write expected total surplus under complete and under incomplete information. First we have

$$\mathbb{E}\left[W^{CI}\right] = \mathbb{E}\left[\Pi^{CI}\right] + \mathbb{E}\left[S^{CI}\right] = \frac{4 - \frac{1}{2^\epsilon}}{(1+\epsilon)(2+\epsilon)(3+\epsilon)}.$$

Second, we have

$$\begin{aligned}\mathbb{E}\left[W^{II}\right] &= \mathbb{E}\left[\Pi^{II}\right] + \mathbb{E}\left[S^{II}\right] = \frac{(4+3\epsilon)^\epsilon}{2^{2\epsilon}(2+\epsilon)^{1+\epsilon}(3+\epsilon)} + \frac{(4+3\epsilon)^{1+\epsilon}}{2^{1+2\epsilon}(1+\epsilon)(3+\epsilon)(2+\epsilon)^{1+\epsilon}} \\ &= \frac{(6+5\epsilon)(4+3\epsilon)^\epsilon}{2^{2\epsilon+1}(1+\epsilon)(3+\epsilon)(2+\epsilon)^{1+\epsilon}}.\end{aligned}$$

Hence,

$$\mathbb{E}\left[W^{CI}\right] > \mathbb{E}\left[W^{II}\right] \Leftrightarrow \frac{4-\frac{1}{2^\epsilon}}{(1+\epsilon)(2+\epsilon)(3+\epsilon)} > \frac{(6+5\epsilon)(4+3\epsilon)^\epsilon}{2^{2\epsilon+1}(1+\epsilon)(3+\epsilon)(2+\epsilon)^{1+\epsilon}} \Leftrightarrow$$

$$4 - \frac{1}{2^\epsilon} > \frac{(6+5\epsilon)(4+3\epsilon)^\epsilon}{2^{2\epsilon+1}(2+\epsilon)^\epsilon} \Leftrightarrow 2^{2\epsilon+1}(2+\epsilon)^\epsilon\left(4 - \frac{1}{2^\epsilon}\right) > (6+5\epsilon)(4+3\epsilon)^\epsilon \Leftrightarrow$$

$$2^{2\epsilon+1}(2+\epsilon)^\epsilon\left(\frac{4(2)^\epsilon - 1}{2^\epsilon}\right) > (6+5\epsilon)(4+3\epsilon)^\epsilon \Leftrightarrow 2^{\epsilon+1}(2+\epsilon)^\epsilon\left(2^{\epsilon+2}-1\right) - (6+5\epsilon)(4+3\epsilon)^\epsilon > 0 \Leftrightarrow$$

$$2^{\epsilon+1}\left(\frac{2+\epsilon}{4+3\epsilon}\right)^\epsilon \frac{2^{\epsilon+2}-1}{6+5\epsilon} > 1 \Leftrightarrow \left(\frac{2(2+\epsilon)}{4+3\epsilon}\right)^\epsilon \frac{2\left(2^{\epsilon+2}-1\right)}{6+5\epsilon} > 1 \Leftrightarrow$$

$$\begin{aligned}&\ln\left[\left(\frac{2(2+\epsilon)}{4+3\epsilon}\right)^\epsilon \frac{2\left(2^{\epsilon+2}-1\right)}{6+5\epsilon}\right] \\ = \;&\epsilon\left[\ln(2) + \ln(2+\epsilon) - \ln(4+3\epsilon)\right] + \ln(2) + \ln\left(2^{\epsilon+2}-1\right) - \ln(6+5\epsilon) \overset{\text{def}}{=} F(\epsilon) > 0.\end{aligned}$$

Note that $F(0) = 0$. Hence we have $\mathbb{E}\left[W^{CI}\right] = \mathbb{E}\left[W^{II}\right]$ for $\epsilon = 0$. Moreover, to prove that $\mathbb{E}\left[W^{CI}\right] > \mathbb{E}\left[W^{II}\right]$ for all $\epsilon > 0$, it suffices to show that $F'(\epsilon) > 0$. Differentiating yields

$$F'(\epsilon) = \ln(2) + \ln(2+\epsilon) - \ln(4+3\epsilon) + \epsilon\left[\frac{1}{2+\epsilon} - \frac{3}{4+3\epsilon}\right] + \frac{2^{\epsilon+2}\ln(2)}{2^{\epsilon+2}-1} - \frac{5}{6+5\epsilon}.$$

Unfortunately it is hard to sign this expression using analytical methods. However, it is easy to verify that $F'(0) = \frac{4\ln(2)}{3} - \frac{5}{6} \approx 0.091 > 0$ and $\lim_{\epsilon\to\infty} F'(\epsilon) = \ln\left(\frac{4}{3}\right) \approx 0.288 > 0$. Moreover, we can plot the graph of $F'(\epsilon)$ (see Fig. 1), which suggests that $F'(\epsilon)$ is indeed strictly positive for all ϵ (with quite a margin).

In order to acknowledge that this is not an analytical proof of the claim, I do not call Result 1 a proposition but a "result." However, there is no doubt in my mind that the claim is true. Indeed, for any given value of the parameter ϵ, one can prove the claim by plugging in this ϵ value in the function $F(\epsilon)$ defined above and check that the number one obtains is positive. □

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